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Boundary String Field Theory as a Field Theory — Mass Spectrum and Interaction

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Abstract: We study the BSFT actions by using an analytic continuation in momentum space. We compute various two- and three- point functions for some low-lying excitations including *massive states* on BPS/non-BPS D-branes. The off-shell two-point functions for the tachyon, the gauge field and the massive fields are found to reproduce the well-known string mass-shell conditions. We compare our action with the tachyon actions previously obtained by the derivative expansion (or the linear tachyon profiles), and find complete agreement. Furthermore, we reproduce the correct on-shell value of the tachyon-tachyon-gauge three-point function on brane-anti-brane systems. Though inclusion of the massive modes has been thought difficult because of the non-renormalizability in string σ models, we overcome this by adopting general off-shell momenta and the analytic continuation.

Keywords: String Field Theories.

Contents

1. Introduction

The outstanding problems in string theory, such as determination of its vacuum, require complete non-perturbative definition of string/M theory. At present, only a few candidates for it are in our hand, among which the most promising formulation may be string field theories [1, 2, 3]. However, their techniques developed for calculating various physical quantities are yet not enough for checking the validity and efficiency of the string field theories. Background-independent (or boundary) string field theory (BSFT) [1] is one of the formulations of the string field theories. For superstrings, the BSFT action S is conjectured to be simply given by the partition function Z of supersymmetric boundary σ models [4, 5, 6]. The BSFT turned out to be very useful in describing exactly some of the important consequences of the

tachyon condensation based on the Sen's conjectures [7] such as tensions of topological defects. Very few have been as successful as the BSFT in extraction of off-shell information in string theory, which suggests that the BSFT is an essential touchstone for exploring non-perturbative aspects of string theories and even obtaining a proper definition of string/M theory.

Though the BSFT is a very attractive formulation, it has some shortcomings to be overcome. One of those is that the definition of the BSFT action given in [1] is rather formal and it is difficult to apply it for actual purposes. Consequently, its relation to worldsheet spectra and string scattering amplitudes is rather unclear. This relation is one of the main points we are going to clarify in this paper. One of the other problems is difficulty in incorporating string massive modes in the BSFT. Since the BSFT action is defined with boundary perturbations (interactions) of the worldsheet σ model action, there is apparent difficulty in defining its partition function for non-renormalizable boundary perturbations. More explicitly, since $\frac{\partial}{\partial\tau}X(\tau)$ has mass dimension one in the boundary one dimensional theory, only the tachyon $T(X)$ and the massless gauge fields $A_\mu(X)$ are renormalizable couplings of the boundary σ model, once perturbative expansion in powers of dimensionless X is employed (this corresponds to Taylor (or derivative) expansion of the space-time fields in the boundary couplings). For this reason, it has been widely believed that only for the tachyon and the gauge field one can compute BSFT actions, but one can not compute the action for massive modes since they correspond to non-renormalizable boundary perturbations, of which the mass dimensions are greater than one. However, there is a loophole in this argument: we may include an infinite number of boundary couplings so that the renormalization is consistent [8]. Although it seems difficult naively to find such a set of bases of infinite dimensions, our answer is quite simple — we use a Fourier decomposition instead of the Taylor expansion, for the boundary interactions. For example, if we express $T(X(\tau))$ in the normal-ordered plane-wave basis as $T(X) = \int dk\ t(k) :e^{ik\cdot X(\tau)}:$, the dimension of this normalized field is less than one for an appropriate region of the momentum value k^2 (this procedure was briefly discussed in appendix A of [9] and similar methods were used in [10, 11]). As we will see in this paper, if we consider $t(k)$ with appropriately negative k^2, the partition function Z is computed to be finite and *there is no need to perform the renormalization further*, thus the BSFT action is well defined for perturbatively all order in $t(k)$. Moreover, we can analytically continue the BSFT action to include $t(k)$ with any k. The important point is that *this procedure can be applied also for the massive fields* and we have a BSFT action for all (bosonic) space-time fields. With use of the normal ordered Fourier basis and the analytic continuation, the BSFT (and also boundary σ models) can incorporate all the excitations of the open string including the massive modes with arbitrary momenta.

In this paper, we study the BSFT action using this analytic continuation method. We compute various two- and three- point functions for some low-lying excitations

(including massive states) on BPS/non-BPS D-branes and brane-anti-branes. In this paper we assume $S_{\rm BSFT} = Z$ [5]. The off-shell two-point functions for the tachyon, the gauge field and the massive fields are found to be reproducing the well-known string mass-shell conditions. We compare our action with the tachyon action previously obtained in [4, 12, 13] by the derivative expansion (or linear tachyon profiles), and find complete agreement for terms existing in both. This is an interesting result since the peturbative tachyon mass was not correctly obtained previously in [4, 14] becasuse of truncation of higher derivative terms in the boundary interaction, while our analytic continuatinuation method includes arbitrary higher drivative terms and gives the correct perturbative tachyon mass. This mechanism has been applied to the tachyon condensation senario in our previous paper [15] in which we have showed that the BSFT linear tachyon backgrounds corresponding to lower-dimensional BPS D-branes and rolling tachyons in fact give rise to deformations of mass-shell structures (expected in effective field theory results such as in [16]) for all the open string excitations, which verifies the Sen's conjectures [7]. One of the other intriguing aspects of our two-point functions is that they seem to require a cut-off for large momenta — this might be the appearance of the minimum length in string theory.

For three-point functions, we reproduce the correct on-shell value of the tachyon-tachyon-gauge three-point function on brane-anti-brane systems. (Most of the other combinations of the fields vanish in the superstring case.) To compute the on-shell value of the three-point function, the limiting procedure empoloyed in [10, 11] is not correct in general. Furthermore, [10] pointed out a problem in which the three-point function with a gauge field exhibits a divergence. We solve these problems by using a field redefintion of the space-time fields which makes the three-point function regular at the on-shell value. The three-point function obtained in this way coincides with the S-matrix computation. As a property of interactions in the BSFT, the following consistency condition is expected [10]: the action $S = Z$ already includes the vertices which reproduce on-shell tree level S-matrix [6], while the path integral of e^{iS} or one-particle-reducible Feynman graphs would give further contributions to it, thus these additional contributions should vanish with on-shell external legs. We have checked this explicitly for our three-point function.

The organization of the paper is as follows. In section 2, after describing our general procedures of computation of the BSFT action, we give two-point functions of spacetime fields including massive excitations reproducing correct superstring spectra for BPS/non-BPS D-branes. A detailed consistency with the tachyon/gauge BSFT actions in the previous literatures is studied. In section 3, a three-point function on a non-BPS brane is obtained and its properties, especially the reproduction of a string scattering amplitude, are studied. section 4 is for discussing various interesting properties of the obtained off-shell action, including an indication of a spacetime resolution in string theory. In appendix A we apply our action to reproduce a part of Sen's result on rolling tachyons [17, 18], and in appendix B, a background gauge

field strength is introduced to see derivative corrections to Born-Infeld actions. In this paper we consider only superstrings since their BSFT action is simple.

2. BSFT action via analytic continuation

After pointing out the difficulty in getting mass-shell conditions in previous techniques (Taylor expansions of the boundary couplings) in the BSFT, in this section we explain how the different choice of the expansion basis consistently gives a BSFT action incorporating the mass-shell conditions. Emphasis is on the fact that, even for massive excitations which are widely believed to be non-renormalizable and so difficult to be treated in boundary σ models, we can successfully write down the BSFT action via the "analytic continuation method". We utilize the momentum dependence of conformal dimensions of off-shell vertex operators to obtain a finite partition function which is the BSFT action. Though one has to make the analytic continuation to get into physically interesting regions of momenta such as the nearly on-shell region, this method turns out to be consistent with all known actions of tachyons and gauge fields derived with Taylor (derivative) expansion of the boundary couplings.

2.1 Brief review of super BSFT action

The disk partition function for a BPS D9-brane is defined by

$$Z = \int DXD\psi \ \exp[-I(X,\psi)] \ . \tag{2.1}$$

In this definition the σ model action I is composed of two terms, $I = I_0 + I_{\mathrm{B}}$, where I_0 is the bulk part*

$$I_0 = \frac{1}{4\pi}\int_\Sigma d^2z\left[\frac{2}{\alpha'}\partial_z X^\mu \partial_{\bar z} X_\mu + \psi^\mu\partial_{\bar z}\psi_\mu + \tilde\psi^\mu\partial_z\tilde\psi_\mu\right] , \tag{2.2}$$

and I_{B} is a boundary interaction which is written by a superfield whose argument is restricted to the boundary,

$$\mathbf{X}^\mu(\tau,\theta) = X^\mu(\tau) + \sqrt{2\alpha'}i\theta\psi^\mu(\tau) \ . \tag{2.3}$$

In this paper we take a convention $\alpha' = 2$. When applied to the context of the BSFT, Σ should not be the upper half plane but a unit disk, and thus $\partial\Sigma$ is not the real axis but a unit circle. So, τ parametrizes the boundary circle of the disk, $-\pi < \tau \leq \pi$. As an example of the boundary perturbation, a massless gauge field on a BPS D-brane is represented by

$$I_{\mathrm{B}} = \int_{\partial\Sigma} d\tau d\theta[-iD_\theta \mathbf{X}^\mu A_\mu(\mathbf{X})] \ , \tag{2.4}$$

*The spacetime metric is diag$(-1, 1, 1, \cdots, 1)$ in this paper.

where the superspace derivative is

$$D_\theta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \tau} \ . \tag{2.5}$$

Other space-time fields can be incorporated as boundary couplings possessing higher powers of D_θ.

For a brane-anti-brane system, we should include the following boundary fermion which represents the Chan-Paton factor of the brane-anti-brane [19]. In the superfield notation, the boundary fermion is

$$\mathbf{\Gamma}(\tau, \theta) = \eta(\tau) + \theta F(\tau) \ , \tag{2.6}$$

and its kinetic action is given by [12, 13]

$$I_{\mathbf{\Gamma}} = \int_{\partial\Sigma} d\tau d\theta \ [-\bar{\mathbf{\Gamma}} D_\theta \mathbf{\Gamma}] = \int_{\partial\Sigma} d\tau \ [\bar{\eta}\dot{\eta} - \bar{F}F] \ . \tag{2.7}$$

This means that η is a propagating boundary fermion while F is an auxiliary field. By the canonical quantization of this action, we may regard $\bar{\eta}, \eta$ and $[\bar{\eta}, \eta]$ as Pauli matrices σ_+, σ_- and σ_3, respectively, which are Chan-Paton degrees of freedom of the two branes. The number of $\mathbf{\Gamma}$'s in the interaction terms is related to the GSO parity. To obtain The disk partition function, we should path-integrate also $\mathbf{\Gamma}$,

$$Z = \int DXD\psi D\eta DF \ \exp[-I(X,\psi) - I_{\mathbf{\Gamma}}(\eta, F)] \ . \tag{2.8}$$

To get a boundary action for a non-BPS D-Brane, we simply restrict $\mathbf{\Gamma}$ to a real superfield. As an example, the tachyon field on the D-brane is represented as

$$\begin{aligned} I_{\mathbf{\Gamma}} + I_{\mathrm{B}} &= \int_{\partial\Sigma} d\tau d\theta \left[-\mathbf{\Gamma} D_\theta \mathbf{\Gamma} + \frac{T(\mathbf{X})}{\sqrt{2\pi}} \mathbf{\Gamma} \right] \\ &= \int_{\partial\Sigma} d\tau \left[\eta\dot{\eta} - F^2 + i\sqrt{\frac{2}{\pi}} \psi^\mu \eta \partial_\mu T(X) - \frac{T(X)}{\sqrt{2\pi}} F \right] \ . \end{aligned}$$

In this boundary action, we can easily integrate out the auxiliary field F to get

$$I_{\mathbf{\Gamma}} + I_{\mathrm{B}} = \int_{-\pi}^{\pi} d\tau \left[\eta\dot{\eta} + i\sqrt{\frac{2}{\pi}} \psi^\mu \eta \partial_\mu T(X) + \frac{1}{8\pi} T(X)^2 \right] . \tag{2.9}$$

When the tachyon field is constant, we immediately obtain the well-known spacetime tachyon potential of the form $e^{-T^2/4}$.

The super BSFT action is conjectured to be given by the disk partition function

$$S_{\mathrm{BSFT}} = Z \ . \tag{2.10}$$

We normalize Z as $Z = \mathcal{T} \int d^{10}x$ when all space-time fields vanish, where $\mathcal{T}$ is the tension of the corresponding brane(s). (Hereafter, we use this normalization of Z.) The validity of the assumption (2.10) has been discussed in [5].

To evaluate the partition function of the open superstring σ model, one needs to specify the explicit form of the boundary couplings. Historically, Taylor expansions for the boundary coupling have been adopted in which for example the tachyon field is expanded as

$$T(X) = T(x) + \partial_\mu T(x) \hat{X}^\mu + \frac{1}{2} \partial_\mu \partial_\nu T(x) \hat{X}^\mu \hat{X}^\nu + \cdots \ , \tag{2.11}$$

where the worldsheet scalar field X^μ is decomposed into its zero mode x^μ plus the oscillator mode $\hat{X}^\mu$. The multiple scalar fields should be normal-ordered so that the boundary coupling is well-defined. The partition function can be evaluated using the worldsheet propagators

$$\left\langle \hat{X}^\mu(\tau) \hat{X}^\nu(0) \right\rangle = -4\eta^{\mu\nu} \log \left| 2 \sin \frac{\tau}{2} \right| \ , \quad \langle \psi^\mu(\tau) \psi^\nu(0) \rangle = \frac{1}{2 \sin \frac{\tau}{2}} \ . \tag{2.12}$$

The propagator for η (in the non-BPS case) is

$$\langle \eta(\tau) \eta(\tau') \rangle = \frac{1}{2} \epsilon(\tau - \tau') = \frac{1}{2} \frac{\sin(\tau/2)}{|\sin(\tau/2)|} \ , \tag{2.13}$$

where $\epsilon(\tau_{12})$ is the sign function. Due to the worldsheet periodicity, the last expression is appropriate, if we define $\frac{\sin(\tau/2)}{|\sin(\tau/2)|} = 0$ for $\tau = 2n\pi$ ($n \in \mathbf{Z}$).

The Taylor expansion (2.11) adopted in most of the literature has provided various interesting results [6, 9, 4, 14, 12, 13], especially on the tachyon condensation and the Sen's conjectures. For the tachyon profiles linear in X, the boundary interaction is path-integrated exactly and gives the tensions of lower-dimensional D-branes. However, this simple expansion (2.11) is obviously incompatible with general mass-shell conditions, because basically the above expansion is a derivative (or α') expansion. Throwing away the higher derivatives to compute the partition function is valid only for nearly massless states, though generic on-shell conditions are apparently different from the massless condition. In this paper, we adopt a different expansion to reconcile the on-shell condition with the computability of the partition function.

2.2 Tachyon two-point function

The essence described in Appendix A of [9] is to use a plane-wave basis instead of the Taylor (derivative) expansion (2.11) to extract the structure of the off-shell tachyon state which can be nearly on-shell. We expand the tachyon field as

$$T(X) = \int dk \ t(k) e^{ik\cdot X} \ = \int dk \ t_{\mathrm{R}}(k) : e^{ik\cdot X} : = \int dk \ t_{\mathrm{R}}(k) \ e^{ik\cdot x} : e^{ik\cdot \hat{X}} : \ . \tag{2.14}$$

The function $t(k)$ is a momentum representation of the tachyon field. We allow generic function $t(k)$ which is thus off-shell. We introduced the normal ordering so that the tachyon off-shell coupling itself is well-defined on the boundary of the worldsheet. The relation to the renormalized tachyon field is

$$t(k) = t_{\rm R}(k) \exp\left[2k^2 \log \epsilon\right] \tag{2.15}$$

where ϵ is the regularization parameter in a regularized version of the propagator (2.12),

$$\left\langle \hat{X}^\mu(\tau)\hat{X}^\nu(0)\right\rangle = 2\eta^{\mu\nu} \sum_{m\neq 0} \frac{1}{|m|} e^{im\tau - |m|\epsilon} \ . \tag{2.16}$$

Accordingly, the renormalized tachyon field in the coordinate representation is

$$T(x) = \exp\left[-2(\log\epsilon)\partial_\mu\partial^\mu\right] T_{\rm R}(x) \ . \tag{2.17}$$

In superstring theories, the open string tachyon appears in unstable D-branes. Here we consider the tachyon in a non-BPS D-branes. The relevant boundary action $I_{\mathbf{\Gamma}} + I_{\rm B}$ is (2.9). Since we work in component fields in the rest of this paper, we refer to (2.9) without the η kinetic term, i.e. the second and the third term in (2.9), as $I_{\rm B}$. The tachyon n-point function is given by

$$\frac{1}{n!}\int dx \ \left\langle (-I_{\rm B})^n \right\rangle \ , \tag{2.18}$$

where $\langle\ \rangle$ denotes the path-integration over $\hat{X}$, ψ and η with the worldsheet action $I_0 + \int d\tau\ \eta\dot{\eta}$ for the non-BPS case. A simplification occurs as explained in [20], the term T^2 in the tachyon coupling $I_{\rm B}$ vanishes in an appropriate region of the momenta, since

$$\begin{aligned}
T(X(\tau))T(X(\tau)) &= \int dk dk'\ t_{\rm R}(k)t_{\rm R}(k')e^{i(k+k')\cdot x} \lim_{\epsilon\to 0} :e^{ik\cdot\hat{X}(\tau+\epsilon)}::e^{ik'\cdot\hat{X}(\tau)}: \\
&= \int dk dk'\ t_{\rm R}(k)t_{\rm R}(k')e^{i(k+k')\cdot x} \lim_{\epsilon\to 0} \exp\left[4k\cdot k' \log|\epsilon|\right] :e^{i(k+k')\cdot\hat{X}(\tau)}: \\
&= \int dk dk'\ t_{\rm R}(k)t_{\rm R}(k')e^{i(k+k')\cdot x} \lim_{\epsilon\to 0} \epsilon^{4k\cdot k'}\ :e^{i(k+k')\cdot X(\tau)}: \\
&= \ 0 \ .
\end{aligned} \tag{2.19}$$

Here we have assumed $k\cdot k' > 0$. Thus we may consider just the term $\psi^\mu\eta\partial_\mu T$ in $I_{\rm B}$ for the computation of the relevant part of the partition function (2.18) and then analytically continue the result to all other region of k. What we are doing here is a kind of an off-shell version of [18].

In this subsection we obtain the tachyon two-point function $\int dx \langle I_{\rm B} I_{\rm B}\rangle$ which provides the mass-shell condition, with use of the boundary propagators (2.12), (2.13).[†] Noting the well-known result

$$\left\langle :e^{ikX}(\tau) :: e^{i\tilde{k}X}(0): \right\rangle = \left| 2\sin\frac{\tau}{2}\right|^{4k\cdot\tilde{k}} e^{i(k+\tilde{k})x} \ , \tag{2.20}$$

we obtain $\langle I_{\rm B} I_{\rm B}\rangle$ as

$$\begin{aligned}
&\int d\tau_1 d\tau_2 \left\langle \left(i\sqrt{\frac{2}{\pi}}\right) :\psi^\mu \eta \partial_\mu T(\tau_1): \left(i\sqrt{\frac{2}{\pi}}\right) :\psi^\nu \eta \partial_\nu T(\tau_2): \right\rangle \\
&\quad = \int dk_1 dk_2 \int d\tau_1 d\tau_2 \left(\frac{-2}{\pi}\right) \frac{1}{2}\epsilon(\tau_{12}) \frac{-1}{2\sin(\tau_{12}/2)} \left|2\sin\frac{\tau_{12}}{2}\right|^{4k_1\cdot k_2} \eta^{\mu\nu} \\
&\qquad \times e^{ik_1\cdot x + ik_2\cdot x} (ik_1)_\mu t_{\rm R}(k_1)(ik_2)_\nu t_{\rm R}(k_2) \\
&\quad = \int dk_1 dk_2\, 2^{4k_1\cdot k_2} \int d\tau_{12} \left|\sin\frac{\tau_{12}}{2}\right|^{4k_1\cdot k_2 - 1} \eta^{\mu\nu} e^{ik_1\cdot x + ik_2\cdot x}(-k_1\cdot k_2) t_{\rm R}(k_1) t_{\rm R}(k_2) \\
&\quad = \int dk_1 dk_2\, e^{ik_1\cdot x + ik_2\cdot x}(-k_1\cdot k_2) t_{\rm R}(k_1) t_{\rm R}(k_2) 2^{4k_1\cdot k_2} 2\sqrt{\pi}\frac{\Gamma(2k_1\cdot k_2)}{\Gamma(2k_1\cdot k_2 + 1/2)} \\
&\quad = -\int dk_1 dk_2\, e^{i(k_1+k_2)\cdot x} t_{\rm R}(k_1) t_{\rm R}(k_2) 2^{4k_1\cdot k_2}\sqrt{\pi}\frac{\Gamma(2k_1\cdot k_2 + 1)}{\Gamma(2k_1\cdot k_2 + 1/2)} \ .
\end{aligned} \tag{2.21}$$

Hence defining

$$Z^{(2)}(y) \equiv 2^{2y}\sqrt{\pi}\frac{\Gamma(y+1)}{\Gamma(y+1/2)} \ , \tag{2.22}$$

we obtain the BSFT action

$$\begin{aligned}
S = Z &= \mathcal{T}\int dx \left[1 - \frac{1}{2}\int dk_1 dk_2\, t_{\rm R}(k_1) e^{ik_1\cdot x} Z^{(2)}(\alpha' k_1\cdot k_2) t_{\rm R}(k_2) e^{ik_2\cdot x} + \mathcal{O}(t_{\rm R}^4)\right] \\
&= \mathcal{T}\int dx \left[1 - \frac{1}{2} T_{\rm R}(x) Z^{(2)}(-\alpha' \overleftarrow{\partial}\cdot\overrightarrow{\partial}) T_{\rm R}(x) + \mathcal{O}(T_{\rm R}^4)\right] \ ,
\end{aligned} \tag{2.23}$$

where we recovered the explicit α' dependence by $\alpha'/2 = 1$. Note that here we haven't performed the target space zero-mode integral $\int dx$, and so we actually compute the BSFT Lagrangian density L. The action has information of all order in its spacetime derivatives but for small magnitude of the tachyon field. We shall see interesting properties of this action below.

First, the tachyon mass-shell condition should be read from the kinetic function $Z^{(2)}$. The equation of motion for the tachyon follows from the action as

$$Z^{(2)}(\partial^2) T_{\rm R}(x) = \mathcal{O}(T_{\rm R}^3) \ . \tag{2.24}$$

[†] After completion of this work, we found that this subsection has an overlapping result with [21].

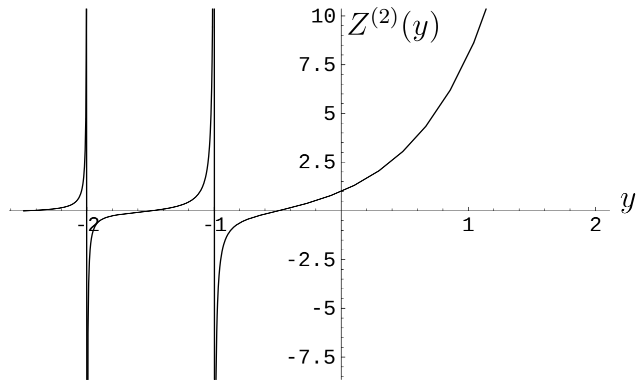

Figure 1: Behavior of the kinetic function $Z^{(2)}(y)$. It has zeros at $y = -n + 1/2$ while poles at $y = -n$ ($n \in \mathbf{Z}^+$). The region $y > -1$ is regular.

Plane-wave solution with an infinitesimal magnitude is represented as $T_{\rm R}(x) = \lambda e^{ik\cdot x}$ ($\lambda \ll 1$), for which the equation of motion reduces to

$$Z^{(2)}(-\alpha' k^2) = 0 \ . \tag{2.25}$$

This can be solved by $k^2 = 1/2\alpha'$, the tachyon mass-shell condition.

Strangely, as seen obviously in Fig. 1, there are an infinite number of solutions for this equation (2.25), $\alpha' k^2 = n + 1/2$ ($n \in \mathbf{Z}_{\geq 0}$). However, poles exists before the next zero is reached from the regular region $y > 0$. These poles might be interpreted as a spacetime resolution or a minimum length in string theory. We shall discuss this point in detail in section 4.

The kinetic operator $Z^{(2)}(y)$ can be expanded around the on-shell momentum, in terms of $y + 1/2$. The coefficient

$$\left[\frac{\partial}{\partial y} Z^{(2)}(y)\right]_{y=-\frac{1}{2}} = \frac{\pi}{2} \tag{2.26}$$

gives us an information on the normalization of the tachyon field once we require the canonical normalization of the kinetic term. (The terms higher in the power of $(y + 1/2)$ in the expansion of $Z^{(2)}(y)$ can be absorbed into the field redefinition of the tachyon field.) This normalization is necessary in comparing the BSFT results with known string scattering amplitudes, which will be studied in section 3.

2.3 Relation to derivative-expanded BSFT tachyon action

We can compare our result (2.23) with the known expression for the tachyon Lagrangians derived so far in BSFT's. An interesting fact is that, although the definition of the partition function naively suggests that a constant T gives vanishing result (2.19) (this is due to the fermion integral $d\theta$ in [18]), the "potential" T^2 term is

reproduced in our result (2.23) after the analytic continuation. In fact, the two-point function in (2.23) has a regular Taylor expansion for small momentum as

$$Z^{(2)}(y) = \sum_{n=0}^{\infty} a_n y^n \ , \tag{2.27}$$

$$a_0 = 1 \ , \quad a_1 = 4\log 2 \ , \quad a_2 = -\frac{\pi^2}{6} + 8(\log 2)^2 \ , \cdots \tag{2.28}$$

and in particular, a_0 is non-vanishing. This means that the Lagrangian is expanded in terms of the derivatives (i.e. slowly-varying field approximation) as

$$\frac{L}{\mathcal{T}} = 1 - \frac{1}{2}T_{\rm R}^2 + 2\alpha' \log 2(\partial_\mu T_{\rm R})^2 + \left(\frac{\pi^2}{12} - 4(\log 2)^2\right) \alpha'^2 (\partial_\mu \partial_\nu T_{\rm R})^2 + \mathcal{O}(\partial^6) \ . \tag{2.29}$$

So there appears $T_{\rm R}^2$ "potential" term. Note that, on the other hand, in the conventional BSFT approaches [4, 13] the $T_{\rm R}^2$ term was due to the expansion of the tachyon potential $e^{-T_{\rm R}^2/4}$ coming from the contact term $T_{\rm R}(X)^2$ in the boundary action which we have neglected as seen in (2.19). Technically speaking, the reason why the potential term $T_{\rm R}^2$ appears is that we have evaluated the worldsheet partition function for generic tachyon momentum k_μ and taken a limit $k \to 0$, which is different from the case of putting $k = 0$ from the first place. We have unawares used an analytic continuation to evaluate the kinetic operator in regions of interest.

A conventional string σ model approach (Taylor expansion or derivative (α') expansion of the boundary couplings) provides a similar form of the effective action for the tachyon [13],

$$\frac{L}{\mathcal{T}} = 1 - \frac{1}{2}\tilde{T}_{\rm R}^2 + 2\alpha' \log 2(\partial_\mu \tilde{T}_{\rm R})^2 + \alpha'^2 \gamma_0 (\partial_\mu \partial_\nu \tilde{T}_{\rm R})^2 + \mathcal{O}(\partial^6) \ . \tag{2.30}$$

The original result in [13] was for a brane-anti-brane, but restricting the complex tachyon field to be real ($\frac{1}{\sqrt{2}}T_{\rm R}^{\rm non-BPS} = {\rm Re}\ T_{\rm R}^{\rm D\overline{D}}$), the action reduces to that of the non-BPS brane as above. The value of the constant γ_0 defined in [13] can be computed[‡] to be $\gamma_0 = \pi^2/12 - 4(\log 2)^2$. This Lagrangian (2.30) completely

[‡]With the following relation for a small ϵ,

$$\sum_{m,r>0} \frac{e^{-(r+m)\epsilon}}{m(m+r)} = \sum_{m,r>0} \int_\epsilon^\infty d\epsilon' \ \frac{e^{-(r+m)\epsilon}}{m} = \int_\epsilon^\infty d\epsilon' \frac{-e^{-\frac{\epsilon'}{2}} \log(1-e^{-\epsilon'})}{1-e^{-\epsilon'}} = \frac{(\log \epsilon)^2}{2} + \frac{\pi^2}{6} - 2(\log 2)^2 + \mathcal{O}(\epsilon)$$

where the summation index m is for positive integers while r is for positive half-odd numbers, γ_0 can be explicitly evaluated as

$$\begin{aligned}\gamma_0 &= -\lim_{\epsilon\to 0} \sum_{m,r>0} \frac{1}{m}\left(\frac{1}{r+m} - \frac{1}{r-m}\right) e^{-(r+m)\epsilon} + \frac{\pi^2}{3} - 4(\log 2)^2 \\ &= -2\lim_{\epsilon\to 0} \sum_{m,r>0} \frac{1}{(m+r)(m-r)} e^{-(r+m)\epsilon} + \frac{\pi^2}{3} - 4(\log 2)^2 = \frac{\pi^2}{12} - 4(\log 2)^2 \ .\end{aligned} \tag{2.31}$$

coincides with our result (2.29). Although it looks that these two Lagrangians were obtained in different regularization schemes, in fact these two regularizations turn out to be the same : the renormalization used in the σ model approach [13] was $T = \tilde{T}_R + \alpha' \log \epsilon \partial_\mu \partial^\mu \tilde{T}_\mathrm{R} + \frac{1}{2}\alpha'^2 (\log \epsilon)^2 (\partial_\mu \partial^\mu)^2 \tilde{T}_\mathrm{R} + \cdots$ with the regularized propagator (2.16), which coincides with the expansion of our regularization (2.17). Therefore we find $T_\mathrm{R} = \tilde{T}_\mathrm{R}$ up to the present order of the derivatives.[§]

Furthermore, we note that our result (2.29) is consistent also with the "usual" BSFT action

$$S = \mathcal{T} \int dx \; e^{-T^2} \mathcal{F}(2\alpha' \partial^\mu T \partial_\mu T) \; , \; \mathcal{F}(x) \equiv x \frac{4^x}{2} \frac{\Gamma(x)^2}{\Gamma(2x)} = 1 + 2(\log 2)x + \mathcal{O}(x^2) \; , \quad (2.32)$$

which was obtained [4] by an exact evaluation of the path-integral with a linear tachyon profile $T = a + u_\mu X^\mu$. Therefore, these different actions (2.23), (2.30) and (2.32) are different expansions of the unique BSFT action which contains correct physical quantities: the tachyon mass, the tachyon potential an so on.

To consider the rolling tachyon solution [17, 18] in the obtained BSFT action (2.23) is interesting. We will study it in appendix A. There we show that a half S-brane marginal deformation of a conformal field theory [22] is actually a solution of our equation of motion with an infinite number of derivatives, preserving the energy while the pressure gradually decreases.

It is noteworthy that for the rolling tachyon solutions we should consider local quantities, such as the Lagrangian density L or energy momentum tensor, instead of the integrated values such as the action $S = \int dx \; L$ [17, 18]. This is because in the latter case the actual action is divergent. It follows that we can not use any partial integration or momentum conservation relations because the integration is divergent for such configuration. Therefore the Lagrangian density should be sensible. Indeed, our analysis above in getting the Lagrangian has not used the partial integration or the momentum conservation relation.

2.4 Gauge field two-point function

The two-point function of the massless gauge fields can be obtained in the same manner. The boundary interaction for the gauge field is (2.4), and written in component fields as

$$I_\mathrm{B} = -i \int_{\partial\Sigma} d\tau \int dk \left(a_\mu(k) \dot{X}^\mu e^{ik_\nu X^\nu} - 2 f_{\mu\nu}(k) e^{ik_\mu X^\mu} \psi^\mu \psi^\nu \right) \; . \quad (2.33)$$

Here $a_\mu(k)$ is the momentum representation of the target space gauge field, and $f_{\mu\nu}(k) \equiv ik_\mu a_\nu(k) - ik_\nu a_\mu(k)$ is that of the field strength. Note that the same action

[§]Note that a change of a renormalization constant, like ϵ_R in [10], corresponds to a field redefinition $T \to e^{c\partial^2} T = T + c\partial^2 T + \cdots$. This gives an extra factor $e^{2c\partial^2}$ to $Z^{(2)}$, which does not change the on-shell condition. Thus it is clearly unphysical.

is used for the gauge fields in a BPS D-brane, a non-BPS D-brane and a brane-anti-brane; the following consideration is applicable to any D-brane system. Let us define the renormalized gauge field for the boundary action to be well-defined,

$$I_{\rm B} = -i \int_{\partial\Sigma} d\tau \int dk \left(a_{{\rm R}\mu}(k) \dot{X}^\mu : e^{ik_\nu X^\nu} : -2 f_{{\rm R}\mu\nu}(k) : e^{ik_\mu X^\mu} : \ \psi^\mu \psi^\nu \right) \ , \quad (2.34)$$

where

$$a_\mu = a_{{\rm R}\mu} \exp\left[2(\log\epsilon) k_\mu k^\mu\right] \ , \quad A_\mu(X) = \exp\left[-2(\log\epsilon)\partial_\mu \partial^\mu\right] A_{{\rm R}\mu}(x) \ . \quad (2.35)$$

Making the operator $\dot{X}^\mu$ normal-ordered with $: e^{ik\cdot X} :$ may produce an additional term,

$$\begin{aligned} &\int d\tau \, \dot{X}^\mu : e^{ik\cdot\hat{X}} : - \int d\tau : \dot{X}^\mu e^{ik\cdot\hat{X}} : \\ &\quad = \int d\tau d\tau_1 d\tau_2 \langle X^\rho(\tau_1) X^\sigma(\tau_2)\rangle \frac{\partial}{\partial\tau} \delta(\tau_1 - \tau)\delta(\tau_2 - \tau)\eta^{\mu\rho} i k^\sigma : e^{ik\cdot\hat{X}} : \ . \quad (2.36) \end{aligned}$$

However, this last expression vanishes with careful treatment with the regularized propagator (2.16). The fermion self-contraction $\psi^\mu\psi^\nu - : \psi^\mu\psi^\nu :$ vanishes by the same reason.¶ So the well-defined boundary interaction for the renormalized gauge field (2.35) is given by (2.34).

Expanding (2.35) to the leading nontrivial order in k, we obtain

$$a_\rho(k) = a_{{\rm R}\rho}(k) + 2k^2(\log\epsilon) a_{{\rm R}\rho}(k) \ . \quad (2.37)$$

Field redefinitions of a_μ which is of the form of a total derivative in the boundary action is still allowed, thus we may add a term to get

$$\begin{aligned} a_\rho(k) &= a_{{\rm R}\rho}(k) + 2k^2(\log\epsilon) a_{{\rm R}\rho}(k) - 2k^\nu k_\rho (\log\epsilon) a_{{\rm R}\nu}(k) \\ &= a_{{\rm R}\rho}(k) - 2ik^\nu (\log\epsilon) f_{{\rm R}\nu\rho}(k) \ . \end{aligned} \quad (2.38)$$

This is the form which has been often used in the σ model approach, see [13]. So our renormalization is the same as that of the σ model.

We would like to evaluate the two-point function of the boundary coupling (2.34), $\int dx \langle I_{\rm B} I_{\rm B}\rangle$, with use of the boundary propagators (2.12). A straightforward calculation shows

$$\begin{aligned} &\langle I_{\rm B} I_{\rm B}\rangle / 2\pi \\ &= \int dk_1 dk_2 \, a_{{\rm R}\mu}(k_1) a_{{\rm R}\nu}(k_2) \int d\tau \left(\frac{\eta^{\mu\nu}}{\sin^2\frac{\tau}{2}} + 4k_2^\mu k_1^\nu \cot^2\frac{\tau}{2} \right) \langle : e^{ik_1 X}(\tau) :: e^{ik_2 X}(0) : \rangle \\ &\quad + \int dk_1 dk_2 \, f_{{\rm R}\mu\nu}(k_1) f_{{\rm R}\rho\sigma}(k_2) \int d\tau \, \frac{\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}}{2\sin^2\frac{\tau}{2}} \langle : e^{ik_1 X}(\tau) :: e^{ik_2 X}(0) : \rangle \ . \end{aligned}$$

¶When there is a background constant field strength, these are non-vanishing. See appendix B.

The remaining correlator is just (2.20), so the integration over τ can be performed and finally gives

$$
\begin{aligned}
\langle I_{\rm B} I_{\rm B}\rangle &= -4\pi^{3/2}\int dk_1 dk_2\ f_{{\rm R}\mu\nu}(k_1) f_{\rm R}^{\mu\nu}(k_2)(1-4k_1\cdot k_2)\frac{\Gamma(2k_1\cdot k_2-1/2)}{\Gamma(2k_1\cdot k_2+1)}e^{i(k_1+k_2)x}2^{4k_1\cdot k_2} \\
&= 8\pi^{3/2}\int dk_1 dk_2\ f_{{\rm R}\mu\nu}(k_1) f_{\rm R}^{\mu\nu}(k_2)\frac{2^{4k_1\cdot k_2}\Gamma(2k_1\cdot k_2+1/2)}{\Gamma(2k_1\cdot k_2+1)}e^{i(k_1+k_2)x} \\
&= 8\pi^2 F_{{\rm R}\mu\nu}(x)\left(1+\frac{\pi^2}{6}(\overleftarrow{\partial}\cdot\overrightarrow{\partial})^2+\mathcal{O}\Big((\overleftarrow{\partial}\cdot\overrightarrow{\partial})^3\Big)\right)F_{\rm R}^{\mu\nu}(x)\ . \qquad (2.39)
\end{aligned}
$$

This incidentally reproduces the well-known fact that the derivative correction of the form $(\partial_\mu f_{\nu\rho})^2$ can have a vanishing coefficient for an appropriate choice of the renormalization condition in the σ model approach. The terms higher order in y can also be put to be zero by a field redefinition since we are dealing with just the two-point function. The situation becomes more nontrivial in a constant field strength background in which one can compare more terms with the results obtained in the other approaches, including string scattering amplitudes. For details see appendix B.

In the present case the gauge invariance ensures that the mass-shell condition is just as expected, $k^2 = 0$. Indeed, if we take the Lorentz gauge $k\cdot a_{\rm R} = 0$ or add an appropriate gauge fixing term to the action (like the $\xi = 1$ (Feynman or Fermi) gauge), and use a partial integration, we can easily show

$$
\langle I_{\rm B} I_{\rm B}\rangle = 8\pi^{3/2}\int dk_1 dk_2\ e^{i(k_1+k_2)x} a_{{\rm R}\mu}(k_1) a_{\rm R}^{\mu}(k_2) Z^{(2)}_{\rm gauge}(2k_1\cdot k_2)\ , \qquad (2.40)
$$

where the kinetic operator is defined as

$$
Z^{(2)}_{\rm gauge}(y) \equiv \frac{2^{2y}\Gamma(y+1/2)}{\Gamma(y)} = \sqrt{\pi}\, y(1+\mathcal{O}(y^2))\ , \qquad (2.41)
$$

which has a zero at $y = -2k^2 = 0$. The kinetic operator is regular for $y > -1/2$ while for negative y there appears periodically zeros and poles. Its similarity to the tachyon kinetic operator (2.22) is obvious. In fact, this structure appears commonly for all the open string excitations as we shall see below for the massive cases. General discussions on this structure will be given in section 4.

2.5 Massive field two-point functions

Now it is clear that one can follow the same procedures to get two-point functions also for the massive excitations. Although the boundary couplings representing the massive excitations are generically non-renormalizable especially when seen in Taylor (derivative) expansions of the boundary couplings, in the normal-ordered plane-wave basis the dimension of the boundary couplings can be chosen to be normalizable for appropriate values of the off-shell momenta. Then the result can be analytically continued to give information even around the on-shell momentum.

2.5.1 First massive state on non-BPS brane

We explicitly show how this mechanism works for the first massive state in a non-BPS D9-brane. The usual quantization on the worldsheet theory results in a single massive antisymmetric two-form field $B_{[\mu\nu]}(x)$ as a physical spectrum. The mass squared is $1/2\alpha'$ and the field is subject to the constraint $\partial^\mu B_{\mu\nu}(x) = 0$. In this subsection we reproduce this result in the BSFT.

In writing the most general boundary couplings, we need a single $\mathbf{\Gamma}$ and two D_θ so that the coupling represents a consistent GSO parity and the mass level:

$$I_{\rm B} = \int d\tau d\theta \Big[D_\theta \mathbf{X}^\mu D_\theta \mathbf{X}^\nu B'_{\mu\nu}(\mathbf{X})\mathbf{\Gamma} + D_\theta D_\theta \mathbf{X}^\mu C_\mu(\mathbf{X})\mathbf{\Gamma} \\ + D_\theta \mathbf{X}^\mu E_\mu(\mathbf{X}) D_\theta \mathbf{\Gamma} + F_\mu(\mathbf{X}) D_\theta D_\theta \mathbf{\Gamma} \Big] \ . \tag{2.42}$$

We can use a partial integration to put $E_\mu = F = 0$, noting that the total derivative vanishes,

$$\int d\tau d\theta \ D_\theta \left[* \right] = 0 \ . \tag{2.43}$$

The resultant Lagrangian has the following gauge symmetry:

$$\delta B'_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \ , \quad \delta C_\mu = 2\Lambda_\mu \ , \quad \delta \mathbf{\Gamma} = -D_\theta X^\nu \Lambda_\nu \ . \tag{2.44}$$

Note that here for this transformation to be a symmetry we ignored $C\Lambda$ and $B\Lambda$. This is justified when we use the normal ordering for the fields and also for the gauge transformation parameter Λ, as in (2.19). Then, if we use the gauge invariant combination $B_{\mu\nu} = B'_{\mu\nu} - \frac{1}{2}(\partial_\mu C_\nu - \partial_\nu C_\mu)$ instead of B', the field C_μ drops out in the action after an appropriate change of $\mathbf{\Gamma}$. Indeed, C_μ can be trivially gauge away and this is an analogue of the Higgs mechanism in the massive sector. Then what we should consider at this level is just

$$\int d\tau d\theta \left[D_\theta \mathbf{X}^\mu D_\theta \mathbf{X}^\nu B_{\mu\nu}(\mathbf{X})\mathbf{\Gamma} \right] \ . \tag{2.45}$$

Note that because $\mathbf{\Gamma}$ transforms by the gauge transformation, the space-time fields which correspond to the GSO odd sector, for example the tachyon, should appear in the gauge transformation law for fields in the GSO even sector. However, these fields come into the transformation as products with Λ, therefore it vanishes for appropriate momentum region (as in (2.19)) and we can neglect this mixing effect in our analytic continuation method at least in the two-point functions.

Let us consider the two-point function of the resultant coupling B. We decompose the fields into component fields with use of (2.3), (2.5) and (2.6). Then,

after integrating out the auxiliary field F, we obtain the following expression for the boundary coupling of the field $B(x)$:

$$I_{\rm B} = 2i\int d\tau \left[\left(2\dot{X}^\mu\psi^\nu B_{\mu\nu}(X) - 4\psi^\mu\psi^\nu\psi^\rho\partial_\rho B_{\mu\nu}(X)\right)\eta\right] \ . \tag{2.46}$$

Our regularization principle is to use the normal ordering for the Fourier transform of the fields,

$$B_{\mu\nu}(X) \equiv \int dk \ b_{{\rm R}\mu\nu}(k) : e^{ik\cdot X} : \ . \tag{2.47}$$

A straightforward calculation shows that

$$\begin{aligned}\langle I_{\rm B} I_{\rm B}\rangle = &\int dk d\widetilde{k} \ e^{i(k+\widetilde{k})\cdot x} 32\pi^{3/2} 2^{4k\cdot\widetilde{k}} \frac{\Gamma(2k\cdot\widetilde{k})}{\Gamma(2k\cdot\widetilde{k}+1/2)} \\ &\times\left[(2k\cdot\widetilde{k}-1/2)b_{{\rm R}\mu\nu}(k)b_{\rm R}^{\mu\nu}(\widetilde{k}) - 4k^\mu b_{{\rm R}\mu\nu}(k)\widetilde{k}_\rho b_{\rm R}^{\rho\nu}(\widetilde{k})\right] \ . \end{aligned}\tag{2.48}$$

The equations of motion are solved by

$$(2k^2+1/2)b_{{\rm R}\mu\nu}(k) = 0, \quad k^\mu b_{{\rm R}\mu\nu}(k) = 0 \ . \tag{2.49}$$

The first equation is the mass-shell condition $k^2 = -1/4$, while the latter equation is a constraint for the $b_{{\rm R}\mu\nu}$ field. These are identical with the worldsheet derivation of the spectrum, thus we have confirmed that our analytic continuation method in the BSFT provides a consistent on-shell information even for massive fields. The number of the physical degrees of freedom is

$$\frac{1}{2}d(d-1) - (d-1) = \frac{1}{2}(d-1)(d-2). \tag{2.50}$$

Note that the constraints give $-(d-1)$ since a constraint $k^\mu b_{{\rm R}\mu 0}$ does not have k_0 component, i.e. a time derivative, and $k^\nu(k^\mu b_{{\rm R}\mu\nu}(k)) = 0$ trivially. So, one of the constraint is not a dynamical one.

The kinetic term (2.48) has a pole at $k^2 = 0$. Therefore it is not well-defined at the zero momentum. This would be related to the fact that the massive modes are non-renormalizable in the Taylor (derivative) expansion of the boundary coupling, since the Taylor expansion is by definition around the zero momentum.

2.5.2 First massive state on BPS D-brane

Let us consider the first massive state on a BPS D-brane. Boundary couplings at this level are composed of three super-derivatives, so we may write down the general couplings as

$$I_{\rm B} = \int d\tau d\theta \left[D_\theta \mathbf{X}^\mu D_\theta \mathbf{X}^\nu D_\theta \mathbf{X}^\rho V_{\mu\nu\rho}(\mathbf{X}) + D_\theta^2 \mathbf{X}^\mu D_\theta \mathbf{X}^\nu W_{\mu\nu}(\mathbf{X}) + D_\theta^3 \mathbf{X}^\mu S_\mu(\mathbf{X})\right] \tag{2.51}$$

Due to the fermionic nature of $D_\theta \mathbf{X}^\mu$, the indices of the field $V_{\mu\nu\rho}$ are totally-antisymmetric. This system has the following two gauge symmetries,

$$
\begin{aligned}
&\text{(a)} \qquad \delta V_{\mu\nu\rho} = \frac{1}{6}\partial_{[\rho}\Lambda_{\mu\nu]}(\mathbf{X}) \ , \quad \delta W_{\mu\nu} = \Lambda_{\mu\nu}(\mathbf{X}) \ , && (2.52)\\
&\text{(b)} \qquad \delta W_{\mu\nu} = \partial_\nu \Lambda_\mu(\mathbf{X}) \ , \quad \delta S_\mu = \Lambda_\mu(\mathbf{X}) \ . && (2.53)
\end{aligned}
$$

Since the indices of the field V are antisymmetric, $\Lambda_{\mu\nu}$ in (a) is also antisymmetric in its indices. Thus we can gauge away the antisymmetric part of the tensor field $W_{\mu\nu}$. Using the second gauge symmetry (b), we may gauge away the field S_μ. Remaining fields are the antisymmetric $V_{\mu\nu\rho}$ and the symmetric part of $W_{\mu\nu}$.

In this subsection, we concentrate on the symmetric field $W_{\mu\nu}$ and shall derive its mass-shell condition by computing its two-point function in the BSFT. (It is easy to find that there is no mixing term among V and W at the level of two-point functions, due to their symmetry property on the indices.) The result should recover the mass-shell condition obtained by the old covariant quantization technique for open superstrings,

$$
k^2 = -\frac{1}{2} \ , \quad \partial^\mu W_{\mu\nu}(x) = 0 \ , \quad W_\mu{}^\mu(x) = 0 \ . \tag{2.54}
$$

That is, the on-shell degrees of freedom of the symmetric tensor field $W_{\mu\nu}$ are traceless and transverse to the momentum, and their mass squared is $1/\alpha'$.

The component expression of the boundary coupling is obtained from the superfield expression (2.51) as

$$
I_{\mathrm{B}} = \int d\tau \left[4\dot{X}^\mu \psi^\nu \psi^\rho \partial_\rho W_{\mu\nu}(X) + (\dot{X}^\mu \dot{X}^\nu - 4\dot{\psi}^\mu \psi^\nu) W_{\mu\nu}(X) \right] \ . \tag{2.55}
$$

We expand the field in the plane-wave basis as before,

$$
W_{\mu\nu}(X) = \int dk \; w_{\mu\nu}(k) e^{ik\cdot X} = \int dk \; w_{\mathrm{R}\mu\nu}(k) : e^{ik\cdot X} : \ . \tag{2.56}
$$

The renormalization due to the normal ordering for the plane wave basis was done with the regularized propagator in the same manner,

$$
w_{\mu\nu}(k) = e^{2k^2 \log \epsilon} w_{\mathrm{R}\mu\nu}(k) \ . \tag{2.57}
$$

However, necessary renormalization is not only this, in contrast to the case of the massless states. Contractions appearing in a part of the boundary couplings generate a finite additional term, since

$$
\begin{aligned}
\dot{X}^\mu \dot{X}^\nu - 4\dot{\psi}^\mu \psi^\nu \ &= \ : \dot{X}^\mu \dot{X}^\nu : -4 : \dot{\psi}^\mu \psi^\nu : + \eta^{\mu\nu} \left(\frac{1}{\sinh^2(\epsilon/2)} - \frac{\cosh(\epsilon/2)}{\sinh^2(\epsilon/2)} \right) \\
&\to \ : \dot{X}^\mu \dot{X}^\nu : -4 : \dot{\psi}^\mu \psi^\nu : -\frac{1}{2}\eta^{\mu\nu} \qquad (\text{as } \epsilon \to 0)
\end{aligned} \tag{2.58}
$$

The cancellation of the divergences here is due to the supersymmetry,[‖] but there remains a finite constant. Because the contractions $\langle \dot{X} X \rangle$ is vanishing, there appears no additional term. We obtain a well-defined boundary couplings with normal ordered operators,

$$I_{\rm B} = \int d\tau \int dk \; w_{{\rm R}\mu\nu}(k) \Big[4ik_\rho : \dot{X}^\mu \psi^\nu \psi^\rho e^{ik\cdot X} : \\ + : \dot{X}^\mu \dot{X}^\nu e^{ik\cdot X} : -4 : \dot{\psi}^\mu \psi^\nu e^{ik\cdot X} : -\frac{1}{2}\eta^{\mu\nu} : e^{ik\cdot X} : \Big] \ . \quad (2.59)$$

The presence of the last term is quite unpleasant: this results in the non-vanishing one-point function, since

$$\langle I_{\rm B} \rangle = \int dk \; {w_{{\rm R}\mu}}^\mu(k) \, e^{ikx} \ . \quad (2.60)$$

The non-zero one-point function immediately means that the vacuum we have chosen is not really a vacuum. However, it is quite unlikely that zero vacuum expectation value for all the excitations is not a consistent open string vacuum. Then what is wrong[**] with this? Our standpoint on this point in this paper is that the one-point function should be put to zero from the first place — there are several natural reasons to believe in this prescription. First, if we closely look at the old covariant quantization, the traceless condition ${w_{{\rm R}\mu}}^\mu = 0$ appears as a physical state condition stemming from a supersymmetry generator $G_{3/2}$ which does not include Lapracian, and this suggests that the traceless condition does not come out as a consequence of the equation of motion of the BSFT. Secondly, let us consider a corresponding procedure in the β-function method in string σ models. In that method, one regards a divergence coming from Wick contractions of higher operators as an equation of motion for the coefficient fields of the remaining operators. In other words, this is a renormalization in the boundary theory. However, in the present case, there is no coefficient field of just $: e^{ik\cdot X} :$ to renormalize the possibly divergent quantity. That is, the coefficient of $: e^{ik\cdot X} :$ should be put to be zero in the β-function method, rather than to be renormalized. In fact, correspondingly to this observation, there is no way to write the $: e^{ik\cdot X} :$ term in a supersymmetric manner, so the term of our concern in the boundary coupling would cause a problem unless put to be zero.

Thus we proceed with assuming the traceless condition ${w_{{\rm R}\mu}}^\mu = 0$. The two-point

[‖] Strictly speaking, the boundary condition of the NS sector we are considering breaks the supersymmetry [6].

[**] In [10], a possibility of cancellation with a higher level boundary term was discussed, but it may not help the cancellation of one-point functions of the whole coupling space.

function is calculated as

$$\langle I_{\rm B} I_{\rm B}\rangle = \int dk d\tilde{k}\; e^{i(k+\tilde{k})\cdot x} w_{{\rm R}\mu\nu}(k) w_{{\rm R}\alpha\beta}(\tilde{k}) \pi\sqrt{\pi} 2^{4k\cdot\tilde{k}} \frac{\Gamma(2k\cdot\tilde{k}-1/2)}{\Gamma(2k\cdot\tilde{k}+1)}$$
$$\times\Big[-8\eta^{\mu\alpha}\eta^{\nu\beta}(2k\cdot\tilde{k})(2k\cdot\tilde{k}-1) - 32\tilde{k}^\mu\tilde{k}^\nu k^\alpha k^\beta + 32\tilde{k}^\mu k^\alpha \eta^{\nu\beta}(2k\cdot\tilde{k})$$
$$+\eta^{\mu\nu}\eta^{\alpha\beta}(2k\cdot\tilde{k}-1/2) + 4(\tilde{k}^\mu\tilde{k}^\nu\eta^{\alpha\beta} + \eta^{\mu\nu}k^\alpha k^\beta)\Big]\ . \tag{2.61}$$

The equations of motion follows as

$$\frac{\Gamma(-2k^2-1/2)}{\Gamma(-2k^2+1)}\Big[-8\eta^{\mu\alpha}\eta^{\nu\beta}(-2k^2)(-2k^2-1) - 32k^\mu k^\nu k^\alpha k^\beta - 32k^\mu k^\alpha\eta^{\nu\beta}(-2k^2)$$
$$+\eta^{\mu\nu}\eta^{\alpha\beta}(-2k^2-1/2) + 4(k^\mu k^\nu\eta^{\alpha\beta} + \eta^{\mu\nu}k^\alpha k^\beta)\Big] w_{{\rm R}\mu\nu}(k) = 0\ . \tag{2.62}$$

Trivial solutions to this equation are

$$2k^2 = 1, 2, 3, 4, \cdots\ . \tag{2.63}$$

We look for nontrivial solutions of the equation (2.62). We first obtain two independent scalar equations by multiplying $k^\alpha k^\beta$ or $\eta^{\alpha\beta}$ on (2.62),

$$\frac{\Gamma(-2k^2-1/2)}{\Gamma(-2k^2)} k^\alpha k^\beta w_{{\rm R}\alpha\beta} = 0\ , \quad \frac{\Gamma(-2k^2-1/2)}{\Gamma(-2k^2+1)}(2k^2+5/2)k^\alpha k^\beta w_{{\rm R}\alpha\beta} = 0 \tag{2.64}$$

Here we have used the traceless condition $w_{{\rm R}\alpha}^{\ \alpha} = 0$. With the momentum different from the trivial solutions (2.63), a unique solution to these equations is

$$k^\alpha k^\beta w_{{\rm R}\alpha\beta} = 0\ . \tag{2.65}$$

Multiplying k^α on (2.62), we obtain

$$\frac{\Gamma(-2k^2-1/2)}{\Gamma(-2k^2-1)} = 0 \qquad \text{or} \qquad k^\alpha w_{{\rm R}\alpha\beta} = 0\ . \tag{2.66}$$

When $k^2 = -1/2$ which solves the first equation, we substitute it back to (2.62) and obtain $k^\alpha w_{{\rm R}\alpha\beta} = 0$. (However when $k^2 = 0$ there appears no additional constraint.) On the other hand, when the second equation $k^\alpha w_{{\rm R}\alpha\beta} = 0$ is satisfied, substituting back it into (2.62), we obtain $k^2 = -1/2, 0, 1/2, 1, \cdots$. So, in sum, we obtain two nontrivial solutions:

$$k^\alpha w_{{\rm R}\alpha\beta} = {w_{{\rm R}\alpha}}^{\alpha} = 0, \quad k^2 = -\frac{1}{2}\ , \tag{2.67}$$

or

$$k^\alpha k^\beta w_{{\rm R}\alpha\beta} = {w_{{\rm R}\alpha}}^{\alpha} = 0, \quad k^2 = 0\ . \tag{2.68}$$

Therefore the solutions of (2.62), different from the trivial solutions (2.63), are (2.67) or (2.68). The first solution (2.67) which is the one with the lowest k^2 recovers the result of the old covariant quantization (2.54). (Note that in the previous cases with lower levels, there similarly appears the additional zeros in the kinetic functions, and the true mass-shell conditions reproducing correctly the worldsheet spectra is the one with the lowest k^2.)

3. Interaction in BSFT — three-point function

Although the two-point functions themselves exhibit interesting structures, for our BSFT to be a field theory, study of nontrivial interactions is indispensable. In this section we compute a three-point function explicitly. We consider only the gauge fields and the tachyon fields for simplicity. For these fields, the three-point function appears in a brane-anti-brane, while there is no three-point interaction for a BPS or a non-BPS D-brane. This follows from the GSO parity and the symmetry $\tau \leftrightarrow -\tau$ in the partition function. The simplest possibility which one might expect in the brane-anti-brane is the $A_\mu^{(-)} T\overline{T}$ three-point function, since it might arise as a part of the covariant derivative of the complex tachyon kinetic term $D_\mu T D^\mu \overline{T}$. In this section, we shall see this in detail.

As for the boundary couplings of tachyon and massless gauge fields on the brane-anti-brane, we follow the notation of [13]. The boundary interaction terms written in component fields after integration of the auxiliary fields are

$$I_{\rm B} = \int_{-\pi}^{\pi} d\tau \left[\frac{i}{2}[\overline{\eta}, \eta] \dot{X}^\mu A_\mu^{(-)}(X) - i[\overline{\eta}, \eta]\psi^\mu\psi^\nu F_{\mu\nu}^{(-)}(X) + i\sqrt{\frac{2}{\pi}}\overline{\eta}\psi^\mu D_\mu T(X) \right.$$
$$\left. -i\sqrt{\frac{2}{\pi}}\psi^\mu \eta D_\mu \overline{T}(X) - \frac{1}{2\pi}\overline{T}(X)T(X) + \frac{i}{2}\dot{X}^\mu A_\mu^{(+)}(X) - i\psi^\mu\psi^\nu F_{\mu\nu}^{(+)}(X) \right] . \quad (3.1)$$

The gauge fields $A^{(\pm)}$ are (plus or minus) linear combinations of the $U(1)$ gauge fields living on two D-branes, $A^{(\pm)} \equiv A_\mu^{(1)} \pm A_\mu^{(2)}$. The complex tachyon field is charged under only the gauge group of $A^{(-)}$, and the covariant derivative is defined as $D_\mu T = \partial_\mu T - i A_\mu^{(-)} T$. We define the operator fields in terms of the normal-ordered Fourier transform as before (in the following, for simplicity we omit the subscript "R" which denotes the renormalized fields),

$$T(X) = \int dk\; t(k) : e^{ik\cdot X} : , \quad \overline{T}(X) = \int dk\; \overline{t}(k) : e^{ik\cdot X} : , \quad (3.2)$$

$$A_\mu(X) = \int dk\; a_\mu(k) : e^{ik\cdot X} : , \quad F_{\mu\nu}(X) = \int dk\; f_{\mu\nu}(k) : e^{ik\cdot X} : \; . \quad (3.3)$$

Note that we defined $\overline{t}(k) = t^*(-k)$. With this renormalized fields, for example the term $T\overline{T}$ in (3.1) vanishes in an appropriate region of the momenta, by the same reason as (2.19) in the non-BPS case. When the momenta are chosen appropriately, any operator product at the same worldsheet point always vanishes, thus the covariant derivatives appearing in (3.1) can be replaced by just the ordinary derivative,

$$D_\mu T(X) = \partial_\mu T(X) \; , \quad D_\mu \overline{T}(X) = \partial_\mu \overline{T}(X) \; . \quad (3.4)$$

So their Fourier transforms are

$$D_\mu T(x) = \int dk\; e^{ik\cdot x} i k_\mu\, t(k) \; , \quad D_\mu \overline{T}(x) = \int dk\; e^{ik\cdot x} i k_\mu\, \overline{t}(k) \; . \quad (3.5)$$

The propagator for η and $\overline{\eta}$ is analogous to (2.13),

$$\langle\eta(\tau)\overline{\eta}(\tau')\rangle = \frac{1}{2}\epsilon(\tau-\tau') = \frac{1}{2}\frac{\sin(\tau/2)}{|\sin(\tau/2)|} \ . \tag{3.6}$$

The computation of the complex tachyon two-point function goes precisely as before, to give

$$\begin{aligned}\langle I_{\rm B}(T)I_{\rm B}(\overline{T})\rangle &= \int d\tau_1 d\tau_2 \ \left\langle \left(i\sqrt{\frac{2}{\pi}}\right) : \overline{\eta}\psi^\mu D_\mu T(\tau_1): \left(-i\sqrt{\frac{2}{\pi}}\right): \psi^\nu \eta D_\nu \overline{T}(\tau_2):\right\rangle \\ &= -\int dk_1 dk_2 \ e^{ik_1\cdot x + ik_2\cdot x} t(k_1)\overline{t}(k_2) 2^{4k_1\cdot k_2}\sqrt{\pi}\frac{\Gamma(2k_1\cdot k_2+1)}{\Gamma(2k_1\cdot k_2+1/2)} \ .\end{aligned} \tag{3.7}$$

The kinetic function is $Z^{(2)}(y)$ (2.22). Expansion for small momenta, $y\sim 0$, coincides with the action obtained in [13].

3.1 Three-point function in brane-anti-brane

It is easy to see that the following three-point functions vanish because some of the fermionic boundary operators do not have their counterpart to be contracted in Wick contractions.

$$\begin{aligned}&TTT = TT\overline{T} = T\overline{T}\overline{T} = TTA^{(+)} = \overline{T}\overline{T}A^{(+)} = TTA^{(-)} = \overline{T}\overline{T}A^{(-)} = TA^{(+)}A^{(+)} \\ &= TA^{(+)}A^{(-)} = TA^{(-)}A^{(-)} = \overline{T}A^{(+)}A^{(+)} = \overline{T}A^{(+)}A^{(-)} = \overline{T}A^{(-)}A^{(-)} = 0 \ .\end{aligned}$$

Most of other three-point functions vanish due to the symmetry $\tau\to-\tau$:

$$T\overline{T}A^{(+)} = A^{(+)}A^{(+)}A^{(+)} = A^{(+)}A^{(+)}A^{(-)} = A^{(+)}A^{(-)}A^{(-)} = A^{(-)}A^{(-)}A^{(-)} = 0$$

except the single one, $T\overline{T}A^{(-)}$. This term is expected, as mentioned earlier. To evaluate this $T\overline{T}A^{(-)}$, first we compute the contribution from the $F^{(-)}$ term in (3.1).

$$\begin{aligned}&\int d\tau_1 d\tau_2 d\tau_3 \left\langle i\sqrt{\frac{2}{\pi}}:\overline{\eta}\psi^\mu D_\mu T(\tau_1):\left(-i\sqrt{\frac{2}{\pi}}\right):\psi^\nu\eta D_\nu\overline{T}(\tau_2): (-i):[\overline{\eta},\eta]\psi^\rho\psi^\sigma F^{(-)}_{\rho\sigma}(\tau_3):\right\rangle \\ &= \frac{-2i}{\pi}\int dk_1 dk_2 dk_3 e^{i(k_1+k_2+k_3)\cdot x} i(k_1)_\mu t(k_1) i(k_2)_\nu \overline{t}(k_2) f_{\rho\sigma}(k_3) \\ &\quad\times\int d\tau_1 d\tau_2 d\tau_3 \left\langle :\overline{\eta}\psi^\mu e^{ik_1\cdot\hat{X}}(\tau_1)::\psi^\nu\eta e^{ik_2\cdot\hat{X}}(\tau_2)::[\overline{\eta},\eta]\psi^\rho\psi^\sigma e^{ik_3\cdot\hat{X}}(\tau_3):\right\rangle \\ &= \frac{-i}{4\pi}\int dk_1 dk_2 dk_3 e^{i(k_1+k_2+k_3)\cdot x} 2^{4(k_1\cdot k_2 + k+2\cdot k_3 + k_3\cdot k_1)}(k_1^\rho k_2^\sigma - k_2^\rho k_1^\sigma) t(k_1)\overline{t}(k_2) f_{\rho\sigma}(k_3) \\ &\quad\times\int d\tau_1 d\tau_2 d\tau_3 \left|\sin\frac{\tau_{12}}{2}\right|^{4k_1\cdot k_2}\left|\sin\frac{\tau_{23}}{2}\right|^{4k_2\cdot k_3 -1}\left|\sin\frac{\tau_{13}}{2}\right|^{4k_1\cdot k_3-1} \\ &= \frac{-i}{\pi}(2\pi)^3\int dk_1 dk_2 dk_3 e^{i(k_1+k_2+k_3)\cdot x}(k_1^\rho k_2^\sigma - k_2^\rho k_1^\sigma) t(k_1)\overline{t}(k_2) f_{\rho\sigma}(k_3) \\ &\quad\times I(2k_1\cdot k_2, 2k_2\cdot k_3 - 1/2, 2k_1 k_3 - 1/2) \ .\end{aligned} \tag{3.8}$$

The integral I was obtained in [11],

$$
\begin{aligned}
I(a_1,a_2,a_3) &\equiv \int_0^{2\pi} \frac{d\tau_1 d\tau_2 d\tau_3}{(2\pi)^3} \left|2\sin\frac{\tau_{12}}{2}\right|^{2a_1} \left|2\sin\frac{\tau_{23}}{2}\right|^{2a_2} \left|2\sin\frac{\tau_{13}}{2}\right|^{2a_3} \\
&= \frac{\Gamma(a_1+a_2+a_3)\Gamma(1+2a_1)\Gamma(1+2a_2)\Gamma(1+2a_3)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+a_1+a_2)\Gamma(1+a_1+a_3)\Gamma(1+a_2+a_3)} \\
&= \frac{2^{2(a_1+a_2+a_3)}}{\sqrt{\pi}^3} \frac{\Gamma(a_1+a_2+a_3)\Gamma(a_1+1/2)\Gamma(a_2+1/2)\Gamma(a_3+1/2)}{\Gamma(1+a_1+a_2)\Gamma(1+a_1+a_3)\Gamma(1+a_2+a_3)} \ . \qquad (3.9)
\end{aligned}
$$

The term coming from $A^{(-)}$ in (3.1) can be evaluated in the same manner,

$$
\begin{aligned}
&\int d\tau_1 d\tau_2 d\tau_3 \left\langle i\sqrt{\frac{2}{\pi}} :\overline{\eta}\psi^\mu D_\mu T(\tau_1): \left(-i\sqrt{\frac{2}{\pi}}\right) :\psi^\nu \eta D_\nu \overline{T}(\tau_2): \frac{i}{2} :[\overline{\eta},\eta]\dot{X}^\rho A_\rho^{(-)}(\tau_3): \right\rangle \\
&\quad = \frac{-i}{\pi} \int dk_1 dk_2 dk_3 e^{i(k_1+k_2+k_3)\cdot x} i(k_1)_\mu t(k_1) i(k_2)_\nu \overline{t}(k_2) a_\rho(k_3) \\
&\qquad \times \int d\tau_1 d\tau_2 d\tau_3 \left\langle :\overline{\eta}\psi^\mu e^{ik_1\cdot\hat{X}}(\tau_1) :: \psi^\nu \eta e^{ik_2\cdot\hat{X}}(\tau_2) :: [\overline{\eta},\eta]\dot{X}^\rho e^{ik_3\cdot\hat{X}}(\tau_3) : \right\rangle \\
&\quad = \frac{1}{2\pi} \int dk_1 dk_2 dk_3 e^{i(k_1+k_2+k_3)\cdot x} k_1\cdot k_2 t(k_1)\overline{t}(k_2) a_\rho(k_3) 2^{4(k_1\cdot k_2+k_2\cdot k_3+k_1\cdot k_3)} \\
&\qquad \times \int d\tau_1 d\tau_2 d\tau_3 \left|\sin\frac{\tau_{12}}{2}\right|^{4k_1\cdot k_2-2} \left|\sin\frac{\tau_{23}}{2}\right|^{4k_2\cdot k_3-1} \left|\sin\frac{\tau_{13}}{2}\right|^{4k_1\cdot k_3-1} \\
&\qquad \times \left(k_1^\rho \sin\frac{\tau_{23}}{2}\cos\frac{\tau_{13}}{2}\sin\frac{\tau_{12}}{2} + k_2^\rho \sin\frac{\tau_{13}}{2}\cos\frac{\tau_{23}}{2}\sin\frac{\tau_{12}}{2}\right) \ . \qquad (3.10)
\end{aligned}
$$

To evaluate the last part, we use the following identity

$$
\begin{aligned}
&\sin A \sin B \cos C \\
&\quad = \frac{1}{4}\left[-\cos(A+B-C) + \cos(-A+B+C) + \cos(A-B+C) - \cos(A+B+C)\right] \ .
\end{aligned}
$$

Then the last integral can be performed to give

$$
\begin{aligned}
&2(2\pi)^2 \int dk_1 dk_2 dk_3 \ e^{i(k_1+k_2+k_3)\cdot x} k_1\cdot k_2 t(k_1)\overline{t}(k_2) \\
&\qquad \times \big[k_1\cdot a(k_3)(-I_1-I_2+I_3) - (k_1 \leftrightarrow k_2)\big], \qquad (3.11)
\end{aligned}
$$

where

$$
I_1 \equiv I\left(\alpha, \beta-\frac{1}{2}, \gamma-\frac{1}{2}\right), I_2 \equiv I\left(\alpha-1, \beta+\frac{1}{2}, \gamma-\frac{1}{2}\right), I_3 \equiv I\left(\alpha-1, \beta-\frac{1}{2}, \gamma+\frac{1}{2}\right),
$$

and $\alpha \equiv 2k_1\cdot k_2$, $\beta \equiv 2k_2\cdot k_3$, $\gamma \equiv 2k_1\cdot k_3$.

Summing up (3.8) and (3.11), we obtain the full three-point function,

$$
\begin{aligned}
&\int dk_1 dk_2 dk_3 \ e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\overline{t}(k_2) \left[-\beta k_1\cdot a(k_3) + \gamma k_2\cdot a(k_3)\right]\mathcal{C} \\
&= \int dk_1 dk_2 dk_3 \ e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\overline{t}(k_2) \\
&\qquad \times \frac{1}{2}\big[(\beta-\gamma)k_3\cdot a(k_3) - (\beta+\gamma)(k_1-k_2)\cdot a(k_3)\big]\mathcal{C} \ , \qquad (3.12)
\end{aligned}
$$

where

$$\mathcal{C} \equiv \frac{\sqrt{\pi}2^{2(\alpha+\beta+\gamma+1/2)}\Gamma(\alpha+\beta+\gamma+1)\Gamma(\alpha+1/2)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+1/2)\Gamma(\alpha+\gamma+1/2)\Gamma(\beta+\gamma+1)} \ . \tag{3.13}$$

In the Lorentz gauge $k_3 \cdot a(k_3) = 0$, the three-point function becomes a rather simple form:

$$\int dk_1 dk_2 dk_3 \ e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\bar{t}(k_2)(k_2-k_1)a(k_3)\frac{\sqrt{\pi}}{2}\frac{\Gamma(\sum_i \alpha_i + \frac{1}{2})\prod_i 4^{\alpha_i}\Gamma(\alpha_i)}{\prod_{i<j}\Gamma(\alpha_i+\alpha_j)} \ , \tag{3.14}$$

where $\alpha_1 \equiv \alpha + 1/2$, $\alpha_2 \equiv \beta$ and $\alpha_3 \equiv \gamma$.

3.2 Exact coincidence with derivative-expanded BSFT

As noted before, the three-point function $T\overline{T}A^{(-)}$ is expected to arise naturally as a part of the tachyon kinetic term with the covariant derivative, $D_\mu T D_\mu \overline{T}$. This is the usual picture considered in [13], while in our case, the covariant derivatives appearing in the boundary interaction vanish due to our normal ordering, (3.5). In this sense, the origins of the $T\overline{T}A^{(-)}$ three-point function are quite different, but we shall see in this subsection that these two methods give the same $T\overline{T}A^{(-)}$ with a suitable field redefinition. In section 2, we have seen nontrivial coincidence on the tachyon potential term T^2 in different approaches. Here is another nontrivial example.

The tachyon two-point function $T\overline{T}$ and the three-point function $T\overline{T}A^{(-)}$ are included in the α'-expanded action obtained in [13] in a σ model approach,

$$\frac{L}{\mathcal{T}} = -T\overline{T} + 8\log 2 D_\mu T D_\mu \overline{T} + 8\gamma_0 D_\mu D_\nu T D_\mu D_\nu \overline{T} + 64i(\log 2)^2 F_{\mu\nu} D_\mu T D_\nu \overline{T} \ . \tag{3.15}$$

Here and in the following study we omit the suffix $(-)$ for simplicity, and neglect terms quartic (or in higher powers) in fields and also terms of $\mathcal{O}(\alpha'^3)$. Relevant terms are expanded explicitly to give the three-point function as

$$D_\mu T D_\mu \overline{T} = \partial_\mu T \partial_\mu \overline{T} - iA_\mu (T\partial_\mu \overline{T} - \partial_\mu T \overline{T}) + \mathcal{O}(A^2) \ , \tag{3.16}$$

$$\begin{aligned} D_\mu D_\nu T D_\mu D_\nu \overline{T} = \ & \partial_\mu \partial_\nu T \partial_\mu \partial_\nu \overline{T} - 2iA_\mu (\partial_\nu T \partial_\nu \partial_\mu \overline{T} - \partial_\nu \partial_\mu T \partial_\nu \overline{T}) \\ & -i(\partial_\nu A_\nu)(T\partial_\nu\partial_\nu T - \partial_\nu \partial_\nu T \overline{T}) + \mathcal{O}(A^2) \ , \end{aligned} \tag{3.17}$$

$$F_{\mu\nu} D_\mu T D_\nu \overline{T} = (\partial_\nu A_\nu - \partial_\nu A_\mu)\partial_\mu T \partial_\nu \overline{T} + \mathcal{O}(A^2) \ . \tag{3.18}$$

We go to the momentum representation of this Lagrangian for our later purpose.

$$S = \mathcal{T}(S_2 + S_3) \ , \tag{3.19}$$

$$S_2 \equiv \int dx \int dk_1 dk_2 \ e^{i(k_1+k_2)\cdot x} t(k_1)\bar{t}(k_2)\left(-1 + 8\log 2 k_1 \cdot k_2 + 8\gamma_0 (k_1 \cdot k_2)^2\right) \ , \tag{3.20}$$

$$\begin{aligned} S_3 \equiv & \int dx \int dk_1 dk_2 dk_3 \ e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\bar{t}(k_2) \\ & \times \Big[k_1 \cdot a(k_3)\left(-8\log 2 + 8\gamma_0(k_1\cdot k_3 + 2k_1\cdot k_2) - 64(\log 2)^2 k_2 \cdot k_3\right) \\ & \quad + k_2 \cdot a(k_3)\left(8\log 2 - 8\gamma_0(k_2\cdot k_3 + 2k_1 \cdot k_2) + 64(\log 2)^2 k_1 \cdot k_3\right)\Big] \ . \end{aligned} \tag{3.21}$$

This Lagrangian should coincide with our BSFT Lagrangian up to a field redefinition. This redefinition should be in higher order in fields, since we already know that our tachyon two-point function coincide with that of [13] without any field redefinition. This means that the redefinition should be of the form $T \to T + TA$. At a glance this redefinition looks strange, in view of that the gauge transformation on T is modified. However, it turns out that this is the case: in our normal-ordered plane-wave basis the tachyon is actually gauge-invariant, because any field multiplication turns out to be vanishing as in (2.19) and thus $e^{i\Lambda}T = T$. (This statement should be understood except for global gauge transformations.) Therefore, what we need as a field redefinition is the one which makes the tachyon field gauge-invariant. To achieve this, we consider the following form of the field redefinition*:

$$t(k_1) \to t(k_1) + \int dk_3 \frac{1}{k_3 \cdot (k_1 - k_3)} a_\mu(k_3)(k_1 - k_3)^\mu t(k_1 - k_3) + \mathcal{O}(a^2) \ , \quad (3.22)$$

$$\bar{t}(k_2) \to \bar{t}(k_2) - \int dk_3 \frac{1}{k_3 \cdot (k_2 - k_3)} a_\mu(k_3)(k_2 - k_3)^\mu \bar{t}(k_2 - k_3) + \mathcal{O}(a^2) \ . \quad (3.23)$$

Substituting this redefinition to the above Lagrangian (3.19), we obtain the following $T\overline{T}A^{(-)}$ terms (we have redefined the momenta as $k_1 - k_3 \to k_1$ and so on so that all the fields have common arguments)

$$\begin{aligned}
&\int dk_1 dk_2 dk_3 \ e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\bar{t}(k_2) \\
&\times \Bigg[k_1 \cdot a(k_3) \Big\{ -8\log 2 + 8\gamma_0(k_1 \cdot k_3 + 2k_1 \cdot k_2) - 64(\log 2)^2 k_2 \cdot k_3 \\
&\qquad + \frac{1}{k_1 \cdot k_3}(-1 - 8\log 2 (k_1 + k_3)\cdot k_2 + 8\gamma_0((k_1+k_3)\cdot k_2)^2) \Big\} \\
&\quad + k_2 \cdot a(k_3) \Big\{ 8\log 2 - 8\gamma_0(k_2 \cdot k_3 + 2k_1 \cdot k_2) + 64(\log 2)^2 k_1 \cdot k_3 \\
&\qquad - \frac{1}{k_2 \cdot k_3}(-1 - 8\log 2(k_2 + k_3)\cdot k_1 + 8\gamma_0((k_2+k_3)\cdot k_1)^2) \Big\} \Bigg] \ . \quad (3.24)
\end{aligned}$$

On the other hand, in terms of small momenta we Laurent-expand $\mathcal{C}$ (3.13) in our BSFT Lagrangian and obtain

$$\begin{aligned}
\mathcal{C} = \frac{1}{3\beta\gamma} \big[& 6 + 12\log 4(\alpha+\beta+\gamma) \\
& + 12(\log 4)^2(\alpha+\beta+\gamma)^2 - \pi^2((\alpha+\beta+\gamma)^2 - 2\beta\gamma) + \mathcal{O}(k^6) \big] \ . \quad (3.25)
\end{aligned}$$

Substituting this expression to the three-point function (3.12), one can see exact coincidence with the one obtained by the field redefinition of the results of [13], (3.24).

*There are other forms satisfying our requirement of the change of the gauge transformation, but (3.22) and (3.23) turn out to be the correct one.

3.3 Reproduction of string scattering amplitude

In this subsection, we shall see that our three-point function reproduces string scattering amplitude at the on-shell momenta. This is a very important check for our claim that the BSFT evaluated with the analytic continuation gives consistent perturbative string theory information, not only the mass-shell conditions.

First we study the value of the three-point function when all the external momenta are set to their on-shell values. The on-shell values of the Lorentz-invariant momentum parameters are

$$\alpha = -1/2, \quad \beta = \gamma = 0 \; . \tag{3.26}$$

We expand the momenta around their on-shell values, $\alpha_1(\equiv \alpha + 1/2), \alpha_2(\equiv \beta), \alpha_3(\equiv \gamma) \ll 1$. The expansion leads

$$\mathcal{C} = \pi \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)}{\alpha_1\alpha_2\alpha_3} + \mathcal{O}(\alpha_i) \; . \tag{3.27}$$

This term gives indefinite result for the on-shell value of the three-point function — the value depends on how one takes the limit to the on-shell momenta. Even worse, the expression may diverge in some limit, for example, $\alpha_2 \sim \alpha_3 \sim \epsilon^2, \alpha_1 \sim \epsilon, \epsilon \to 0$. However, we can subtract a part of this term by a field redefinition of the tachyon, to make it definite. The redefinition for the tachyon is of the form same as that of the previous subsection. After subtracting these indefinite terms, we arrive at an expression which is irrelevant on how is the limit to the on-shell momenta.

Let us see this in detail. The above limiting behavior in $\mathcal{C}$ results in the three-point function in the Lorentz gauge as

$$\begin{aligned}\int dk_1 dk_2 dk_3 \; & e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\bar{t}(k_2) \\ & \times \frac{-\pi}{2}(k_1-k_2)\cdot a(k_3) \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}{\alpha_1\alpha_2\alpha_3} + \cdots \; .\end{aligned} \tag{3.28}$$

The indefinite part can be cast into the form

$$\frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}{\alpha_1\alpha_2\alpha_3} = \frac{\alpha_1+\alpha_2}{\alpha_3} + \frac{\alpha_1+\alpha_3}{\alpha_2} + \frac{\alpha_2+\alpha_3}{\alpha_1} + 2 \; . \tag{3.29}$$

The first term can be eliminated by a field redefinition of the form similar to (3.22) and the second by (3.23), while the third one can be eliminated by a field redefinition of the gauge field of the form

$$a_\mu(k_3) \to a_\mu(k_3) + \int dk \; \frac{f(k,k_3)}{4k\cdot(k_3-k)+1} t(k_3-k)\bar{t}(k) \tag{3.30}$$

where f is chosen in such a way that the resulting term coming out of the gauge kinetic term can eliminate the third term in (3.29). (The denominator corresponds

to the factor $1/\alpha_1$.)* The last term in (3.29) is remaining and cannot be absorbed into the field redefinition, because any redefinition should not be singular at on-shell momenta and so for example a redefinition $a_\mu \to a_\mu + \int \frac{1}{\alpha_2+\alpha_3} a_\mu t$ is not allowed ($\alpha_2 + \alpha_3 = -2k_3^2$ is included in the kinetic term). Thus the on-shell three-point function reads with the last term in (3.29) as

$$L_3 = 2\mathcal{T}_{\rm D9} N_T^2 N_A (-\pi) \int dk_1 dk_2 dk_3 \; e^{i(k_1+k_2+k_3)\cdot x} t(k_1)\bar{t}(k_2) \; (k_1-k_2)^\mu a_\mu^{(-)}(k_3) \; . \tag{3.31}$$

Here $2\mathcal{T}_{\rm D9}$ is the tension of the brane-anti-brane, and we have newly included the normalization factors for the tachyon and the gauge field N_T and N_A respectively in the boundary coupling: $T(x) \to N_T T(x)$, $A_\mu^{(\pm)} \to N_A A_\mu^{(\pm)}$. We have put $N_T = N_A = 1$ in the calculations so far, but we need these hereafter to compare our result with a string scattering amplitude. The normalization factors $N_{T,A}$ can be fixed by requiring the canonical normalization for the kinetic term (two-point functions). The tachyon kinetic term can be expanded as (see (2.26))

$$2\mathcal{T}_{\rm D9} N_T^2 T(x) \left(\pi \left[\overleftarrow{\partial} \cdot \overrightarrow{\partial} - \frac{1}{4} \right] + \mathcal{O}\left(\left[\overleftarrow{\partial} \cdot \overrightarrow{\partial} - \frac{1}{4} \right]^3 \right) \right) \overline{T} \; . \tag{3.32}$$

Thus, up to a higher order field redefinition of the form

$$t(k) \to \left(1 + \mathcal{O}\left((k^2+1/4)^2\right)\right) t(k) \; , \tag{3.33}$$

the canonical normalization of the kinetic term ($L = |\partial_\mu T|^2 - \frac{1}{4}|T|^2$) implies

$$2\mathcal{T}_{\rm D9} N_T^2 \pi = 1 \; . \tag{3.34}$$

In the same manner, we obtain the gauge two-point function for the present brane-anti-brane case†

$$\begin{aligned} 2\mathcal{T}_{\rm D9} \frac{1}{2} \left\langle I_{\rm B}(A^{(-)}) I_{\rm B}(A^{(-)}) \right\rangle = 2\mathcal{T}_{\rm D9} N_A^2 \pi^2 F_{\mu\nu}^{(-)}(x) F^{(-)\mu\nu}(x) \\ +(\text{higher derivatives}) \end{aligned} \tag{3.35}$$

with a similar expression also for $A_\mu^{(+)}$. This gives the following normalization relation so that $L = \frac{1}{4}(F_{\mu\nu}^{(1)})^2 + \frac{1}{4}(F_{\mu\nu}^{(2)})^2$ is achieved,‡

$$2\mathcal{T}_{\rm D9} N_A^2 \pi^2 = \frac{1}{8} \; . \tag{3.36}$$

*It is noteworthy that the same field redefinition simultaneously removes the divergence of the three-point function at zero momenta, because the field redefinition of the gauge field (3.30) is regular at the vanishing momenta.

†To obtain the expression (3.35), we have to multiply $\langle :\overline{\eta}\eta(\tau_1): :\overline{\eta}\eta(\tau_2): \rangle = 1/4$ on the previous result for the BPS D-brane (2.39). The resultant normalization in (3.35) coincides with the result of the σ model calculation in [13].

‡We have implicitly used the same normalizations for $A^{(+)}$ and $A^{(-)}$, because otherwise these canonical kinetic terms for $A^{(1,2)}$ would not be achieved.

Since we know the tension of the D9-brane, the normalizations N_T and N_A are completely fixed[§] from (3.34) and (3.36), and thus the normalization of the three-point function (3.31) is determined.

We may compare this normalized three-point function (3.31) with the known tree-level string scattering amplitude [23],

$$e(2\pi)^{10}\delta\left(\sum k_i\right)(k_1-k_2)\cdot\zeta^{(-)} \ , \tag{3.37}$$

where $\zeta^{(-)}_\mu$ is the polarization of the gauge field $A^{(-)}_\mu$, and e is the open string coupling defined in [23]. We can deduce

$$e = N_A \tag{3.38}$$

by just looking at the structure of the covariant derivatives in ours and [23]. In [23], the constant Wilson line was introduced as a background which shifts the tachyon momentum as $k_\mu - eA^{(-)\text{b.g.}}_\mu$, while in our case the change of the normalization of the gauge field $A^{(-)}_\mu \to N_A A^{(-)}_\mu$ gives rise to the change of the covariant derivative to $\partial_\mu - iN_A A^{(-)}_\mu$. Substituting this relation (3.38) and the tachyon normalization (3.34) into our three-point function (3.31), we find the exact reproduction of the scattering amplitude (3.37). Thus we conclude that the BSFT three-point function (3.31) in which the divergent part has been subtracted by a field redefinition coincides with the string scattering amplitude.[¶]

[§]Since the tension is of order $1/g^2_{\text{open}} \sim 1/g_{\text{closed}}$, the determined normalizations $N_{T,A}$ are of order g_{open}, which is consistent. To determine the exact value of $N_{T,A}$ in terms of g_{open}, we need the explicit expression for the D-brane tension written by the open string coupling defined in our boundary couplings. This can be obtained by a computation of a one-loop amplitude of an open string normalized in our convention and using the open-closed duality.

[¶]An amusing point is that a naive comparison of our boundary couplings with the string vertex operators gives the relation same as (3.38). The vertex operators in the 0 picture in [23] were

$$\mathcal{V}_T = -4e\, c\, k_\mu\psi^\mu e^{ik\cdot X}\cdot\frac{1}{2}(\sigma_1 \pm i\sigma_2) \ , \tag{3.39}$$

$$\mathcal{V}_A = ie\, c\,\zeta_\mu\left(\dot{X}^\mu - 4i\psi^\mu k_\nu\psi^\nu\right)e^{ik\cdot X}\cdot\frac{1}{2}(\sigma_0 \pm \sigma_3) \ , \tag{3.40}$$

where c is the worldsheet ghost and the Pauli matrices σ's are Chan-Paton factors ($\sigma_0 \equiv 1_{2\times 2}$). Comparing these with our boundary couplings by simply dropping the ghost, we obtain

$$-N_T\sqrt{\frac{2}{\pi}} = -4e \ , \quad N_A = e \ . \tag{3.41}$$

The second relation is precisely what we learned in comparison of the covariant derivatives (that is, the gauge transformation laws after the field redefinition). The ratio of the relations (3.41) is consistent with the ratio of our results (3.34) and (3.36). Note that the normalization of the vertex operators (3.39) and (3.40) was fixed in [23] by demanding the unitarity (this is the reason why we had to normalize our BSFT two-point function canonically, to compare our result with the scattering amplitude). This suggests that our BSFT is automatically unitary by construction. However, in this comparison of the vertex operators, it is not clear why we may simply drop the ghost.

Note that in the bosonic case [10] the tachyon three-point function with on-shell momenta was evaluated by taking the limit symmetric under the momentum exchange among the three tachyons. However, with a careful evaluation, this turns out to be unnecessary – one can in fact show that the three-point function given in [11] is free of indefiniteness around the on-shell momenta. So the value of the bosonic tachyon three-point function does not depend on how one takes the on-shell limit, and is definite without any field-redefinition.

Second, let us see what happens to the three-point functions when one of the fields is set to their on-shell values. When the tachyon T (or $\overline{T}$) is on-shell and thus $\alpha_1+\alpha_3=0$ (or $\alpha_1+\alpha_2=0$ respectively), the factor $\mathcal{C}$ vanishes and so the three-point function disappears. When the gauge field is on-shell ($\alpha_2+\alpha_3=0$ and $k_3\cdot a(k_3)=0$), although $\mathcal{C}$ is non-vanishing, the coefficient in front of $\mathcal{C}$ in (3.12) vanishes. Therefore, when one of the three outer legs is on-shell, the three-point function vanishes. This results in the vanishing of on-shell one-particle-reducible diagrams, when we treat the BSFT action as a field theory action and perform the usual Feynman rule for getting higher point functions. In fact, although the propagator connecting the vertices is diverging at the on-shell momentum, it is not powerful enough to cancel the vanishing of the on-shell vertices connected to two boundary points of the propagator in the Feynman graph. This is quite satisfactory since, as mentioned in the introduction, the BSFT action $S_{\rm BSFT}=Z$ already includes the vertices which reproduce the on-shell tree level S-matrix [6], and one-particle-reducible Feynman graphs generated by lower vertices should vanish.[‖]

4. Generalities — minimum length in string theory

In this section we consider general properties of the super BSFT action. If we look at the structure of the worldsheet boundary integral, it is easy to notice that for massive fields with the mass-shell condition $k^2=-(N-1)/2\alpha'$, the integral is convergent for

$$2\alpha' k\cdot\tilde{k} > N\ , \tag{4.1}$$

where N is the oscillator level of the open string excitations. In our boundary couplings, N is the number of the derivative D_θ. This can be seen in the the dimensions of the corresponding operators. So, for two-point functions in BSFT, there is a region for the momentum where the integral over the worldsheet boundary is well-defined and finite. This is interesting in view of the fact that usually in BSFT the massive excitations are believed not to be treatable because they are non-renormalizable operators. This convergent region (4.1) of the momentum can apply also for three- or more point functions, since the divergence appears only when two of the vertices

[‖]The paper [10] considered a bosonic case in which there are corrections containing the beta function to the relation $S_{\rm BSFT}=Z$. In our superstring case, there is no such correction.

collide to each other. Therefore, if any pair of the momenta satisfies (4.1) the integral is finite. Then we can use the analytic continuation to obtain the BSFT action for any momenta.

However, there are many singularities in the momentum space. For massive excitations, the two-point functions are expected to have the following form:

$$Z^{(2)}(y) \sim \frac{2^{2y}\Gamma(y+(2-N)/2)}{\Gamma(y+(1-N)/2)} \ , \tag{4.2}$$

where $y = -\alpha' \overleftarrow{\partial} \cdot \overrightarrow{\partial} = -\alpha' k^2$. All of our BSFT results are of this form** — the tachyon ($N = 0$) $Z^{(2)}(y)$ is exactly above, and the gauge field ($N = 1$) kinetic function takes this form in the Lorentz gauge $k^\mu a_\mu = 0$, which is also the case for the massive fields. From this expression, we observe that there is a common structure in the kinetic term of the open string excitations — an infinite number of poles and zeros appear in the tachyonic region ($k^2 > 0$) in the kinetic operator $Z^{(2)}$. For the tachyon, see Fig. 1. Note that for $k^2 > -(N-1)/2\alpha'$ the operator with the level N is irrelevant and the first pole is found at $k^2 = -(N-2)/2\alpha'$. This means that we can make an analytic continuation of the two-point function $Z^{(2)}$ to the inside of the irrelevant region until hitting the pole.

This quite intriguing pole/zero structure might suggest a minimum length in space which can be observed by fundamental strings. Let us regard the BSFT action as an off-shell generalization of the tree level S-matrix generating effective action as in [6]. Near the on-shell momentum k, we may try to redefine the fields so that they have canonically normalized kinetic terms. For example, the redefinition of the gauge field should be $A'_\mu = (Z^{(2)}_{\rm gauge}(\alpha'\partial^2))^{1/2} A_\mu$. However, $Z^{(2)}_{\rm gauge}(y)$ (2.41) has a singularity at $y = -1/2$. Moreover, it changes the sign when we go over the singularity, which implies that the field redefinition becomes imaginary, then is not allowed. Therefore we might have to restrict the region of the momentum k in order to avoid the singularity,†† at least if we naively interpret the two-point functions, derived by using a coordinate system of the space of the boundary couplings (spacetime fields) suggested in BSFT. The introduction of the upper-bound for the momentum k^2, somewhat like a Briroinn zone, implies that the space becomes effectively discretized with the scale $\sqrt{\alpha'}$, that is, the string length. One can interpret this as a spacetime resolution, or rather to say, the minimum length in string theory.

On the other hand, one can in principle compute the effective action using cubic string filed theory [2] at least for the bosonic case. This should coincide with our BSFT action upto field redefinition ambiguities. For on-shell fields, these two are considered to be the same as described in [11] for bosonic three-point tachyons and

**For a related discussion, see [10].

††Here we assume our analytic continuation is valid. We note that the tachyon kink solution $T = u_9 X^9$, for example, has an expression $t(k) = iu_9 \frac{\partial}{\partial k}\delta(k)$ in the momentum space, which is within the allowed momentum region even though it is an off-shell configuration.

as agrgued in [6]. The problem is, as mentioned above, that the field redefition used for relating $Z^{(2)}$ to the standard kinetic terms in the cubic string field theory does not exist at and beyond the singularity of $Z^{(2)}$. Furthermore, if we remember the Witten-Shatashvili formula [1] $(\delta/\delta g_i)S_{\rm BSFT} \sim \beta^i G_{ij}$ (where β^i is the beta function for the boundary coupling (spacetime field) g_i, and G_{ij} is the metric of the space of the couplings) and $\beta^i \sim (k^2 + (N-1)/2\alpha')$, we find that the extra poles and zeros are coming from the metric G_{ij} as was pointed out in [11]. This indicates that the problem is not only for the kinetic terms, but a more general one.

One possible resolution is that the cubic string field theory may also have the restriction on the momentum for some reason. However, this is not likely because non-singular field redefinition does not relate interaction terms to the kinetic term at tree level. Another possibility is that the coordinate system of the BSFT is singular at the singularity though the physics is not singular and we should use another appropriate coordinate system of the space of the boundary couplings beyond the singularity. Since the fields of the BSFT is in some sense natural, it is reasonable to expect that the singularity of the coordinate system means some peculiar physics appearing there, like the example of the noncommutative soliton discussed by Sen [24]. Hence the singularity of the space of the boundary couplings may reflect some kind of the minimum length.

Meanwhile, let us regard the BSFT action as that of a constructive (string) field theory [1], instead of taking it as an effective action. Then in this interpretation we would have to path-integrate the fields, which causes a serious problem for loop amplitudes. Although the loop amplitudes of the BSFT are known to be difficult to deal with and we do not have any definite answer to that, we would like to give a few comments on it. The problem is that we have to perform an integration over the loop momenta in perturbative evaluation of the loop amplitudes. It seems that the momentum bound which we studied in this section is an obstacle to perform the loop momentum integration. Furthermore, on internal vertices, the momentum conservation cannot satisfy the lower bound for the momenta. To avoid this problem, we can decompose the propagator obtained in the BSFT as

$$\frac{\Gamma(y)}{\Gamma(y+1/2)} = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{y+n} = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\frac{1}{y+n}\ , \tag{4.3}$$

which is a sum of a massless propagator and infinitely many tachyonic propagators. (This formula is similar to the one used in [25] but differs in that the poles appear in the tachyonic side in our case.) Then effectively there appears no singularity and the bound may disappear, at the sacrifice of introduction of infinitely many tachyons. Using a Feynman rule with these propagators and vertices of the BSFT action, it might be possible to reproduce correct string amplitudes in a field-theoretical manner, also for the loop amplitudes.

There is another way to avoid this problem on the loops. In the previous section, we found that the three-point function disappear once one of the three momentum legs is put to be on-shell. If this is the general feature of the n-point functions in the BSFT action evaluated with the disk partition function, any field-theoretical loop amplitude constructed from these vertices vanishes. Thus from the first place there is no problem concerning the cut-off of the loop momenta. Then, how can we reproduce the string theory loop amplitudes from the BSFT? A possible answer to this question might be that we have to consider also the BSFT action based on partition functions of higher genus worldsheets. (This standpoint is different from Witten's original proposal that the disk partition function is a definition of the BSFT.) There are several problems even for one-loop BSFT's [26], which is beyond the study in this paper.

5. Conclusion and discussions

The main virtue of the analytic continuation method which we have developed in this paper is that it accommodates the BSFT and string σ models to the massive excitations. Allowing the normal-ordered Fourier basis for the boundary couplings of the σ model partition function (= the BSFT action), we can choose appropriate momentum which manifestly makes the partition function finite. The analytic continuation of the momenta brings us to any region of interest, including especially the on-shell momenta. The resulting BSFT off-shell two-point functions for the tachyon field, the gauge field and some of the massive fields on a BPS/non-BPS D-brane reproduce the well-known string mass-shell conditions. The BSFT three-point function computed for two tachyons and a single gauge field on a brane-anti-brane provides the correct on-shell value of the standard scattering amplitude calculation. In doing this, we have used the field redefinition of the tachyon and the gauge field which makes the three-point function regular at the on-shell value, instead of the prescriptions in [10, 11]. The three-point function obtained in section 3 is consistent with the assumption $S_{\rm BSFT} = Z$: since the partition function already includes all the on-shell scattering amplitudes, the BSFT action should not generate additional contributions to these from one-particle-reducible Feynman graphs. In fact, our vertex vanishes if we put one of the external legs to be on-shell.

We have found that our BSFT tachyon action coincides with those constructed so far in the Taylor (or derivative) expansion of the tachyon boundary coupling or in the linear tachyon profiles. As for the two-point functions we need no field redefinition to relate these, since the standard renormalization of the fields in the σ model turns out to be identical with our renormalization based on the normal-ordered Fourier basis. For the three-point function, a certain field redefinition obtained by looking at the difference in gauge transformations on both sides gives a perfect agreement of the on-shell tachyon-tachyon-gauge interaction.

Because our analysis is based on the perturbation of the σ model couplings for $T, A_\mu, \cdots$, how our methods may incorporate non-perturbative effects of BSFT is a quite important subject to study. To illustrate this, let us recall the example [18] where a perturbative series in the BSFT sums up to give a neat result. For the rolling tachyon solution $T(X) = \lambda e^{-X_0}$ the partition function was obtained perturbatively as $Z = \sum_n (-(T(X_0))^2)^n/n$, which can be analytically continued to $1/(1+T^2)$ to give the information of the final state of the rolling tachyon [18]. This suggests that we need anyway the analytic continuation in general for the σ model couplings. Then, how we can see directly non-perturbative effects of BSFT, like instantons in gauge theories? An answer is found in the evaluation of the partition function with the linear tachyon profiles, or the constant gauge field strength, or more general cases considered in [27], which can be regarded as non-perturbative results. The path integral of the world sheet action with these profiles reduces to a Gaussian integral, and is evaluated exactly, thus the results are non-perturbative in the boundary couplings. To find a more universal relation between this and the analytic continuation method, and how to extend our analysis beyond the perturbation, are interesting questions.

We would like to point out an intriguing similarity between our tachyon two-point function (2.23) and the partition function of the linear tachyon profile (2.32). In fact, $\mathcal{F}(x)$ in (2.32) is written as $2\sqrt{\pi}\Gamma(x+1)/\Gamma(x+1/2)$ which, as a function, is the same as $Z^{(2)}(y)$ defined in (2.22) if we absorb the 2^{2y} factor into the redefinition of the tachyon field. In the former, the argument is $x = 2\alpha'(\partial_\mu T)^2$ while in the latter y is just $\alpha'\partial^2$. This surprising similarity suggests that there might be some relation between these two, such as a certain field redefinition. If this is true, our result could be applied to the analysis of the rolling tachyon physics using BSFT [28, 15].

Although our approach have reproduced various perturbation results of string theory, there are still many indispensable aspects which need to be explored to have a definite "field-theoretical" BSFT. First, we need to include space-time fermions. This is difficult because we use the NS-R formalism, but it might be overcome by taking into account the broken supersymmetry [29]. Secondly, general n-point functions may have more complicated singularity structures, such as cuts. We do not have tools powerful enough to compute explicitly the higher-point functions which might exhibit interesting structures. At least, the four-point functions should reproduce the Veneziano amplitude and the s-t channel duality. Lastly, the most important is to understand the loop amplitudes of the BSFT discussed in the previous section. These are very interesting questions and we hope to come back to these in the future.

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A. Rolling tachyon solution

In this appendix, we study the properties of a classical solution of the tachyon equation of motion (2.25). Consider a solution of a plane wave with the momentum $k_0 = i/\sqrt{2\alpha'}, k_i = 0$, which is well promoted to be an exactly marginal operator [22, 18], called a half S-brane. For this tachyon profile $T = \lambda e^{x^0/\sqrt{2\alpha'}}$, the energy and the pressure were computed in [18] with an arbitrary λ, by evaluating a disk amplitude with an insertion of a single closed string vertex. Here we may compute those observables by purely field-theoretical method in the sense of target space, since we have an off-shell action for the tachyon. Let us see how our action reproduce a part of the results of [18]. We couple the system to the gravity in a natural manner,

$$S = \mathcal{T} \int dx \sqrt{-g} \left[1 - \frac{1}{2} T(x) Z^{(2)}(-\overleftarrow{\nabla}_\mu \alpha' g^{\mu\nu} \overrightarrow{\nabla}_\nu) T(x) + \mathcal{O}(T^4) \right] . \tag{A.1}$$

From this expression the energy is defined as $T_{00} = [2\delta L/\delta g^{00}]_{g^{00}=-1}$. We may make a dimensional reduction to 1 dimension without losing generality. The reduced Lagrangian has an infinite number of derivatives (we define $\sqrt{-g^{00}} \equiv v$),

$$L = \mathcal{T} \frac{1}{v} \left[1 - \frac{1}{2} \sum_{n=0}^{\infty} a_n \alpha'^n v^{2n} (\nabla_0^n T)^2 \right] , \tag{A.2}$$

thus it is nontrivial that the solution $T = \lambda e^{x^0/\sqrt{2\alpha'}}$ has a conserved energy, which we are going to check. The covariant derivatives are explicitly written as $\nabla_0^n T = (1/v^{n-1}) \partial_0 (v\partial_0 (v\partial_0 (\cdots (v\partial_0 T))))$. Let us take a differentiation of L with respect to v, giving the energy in the system,

$$\begin{aligned} \frac{1}{\mathcal{T}} \frac{\delta L}{\delta v}\bigg|_{v=1} &= -\left(1 - \frac{1}{2} a_0 T^2\right) - \frac{1}{2} \sum_{n=1}^{\infty} a_n \alpha'^n \frac{\delta}{\delta v} \left[\frac{1}{v} \overbrace{(v\partial_0 (v\partial_0 (v\partial_0 (\cdots (v\partial_0}^{n} T))))^2 \right]_{v=1} \\ &= -\left(1 - \frac{1}{2} a_0 T^2\right) - \frac{1}{2} \sum_{n=1}^{\infty} a_n \alpha'^n \sum_{i=1}^{2n-1} (\partial_0^i T)(\partial_0^{2n-i} T)(-1)^{i-n} . \end{aligned} \tag{A.3}$$

The equation of motion $\sum_{n=0}^{\infty} a_n (-1/2)^n = 0$ can be used under the substitution of the solution $T = \lambda e^{x^0/\sqrt{2\alpha'}}$. We finally obtain the conserved energy

$$T_{00} = \mathcal{T} , \tag{A.4}$$

as we expected.

It is easy to show that the pressure T_{ii}, which can be computed in the same manner from our Lagrangian, is identical to the partition function itself, since we are dealing with a spatially homogeneous rolling. This pressure reproduces the small field expansion of the result of the worldsheet computation [18],

$$T_{ii} = \mathcal{T}\left(1 - \frac{\pi}{2}\left(\lambda e^{x^0/\sqrt{2\alpha'}}\right)^2 + \mathcal{O}(\lambda^4)\right) . \tag{A.5}$$

We have used a relation $Z^{(2)}(y = 1/2) = \sum_n a_n (1/2)^n = \pi$. The pressure is decreasing, and the system is decaying to the tachyon matter [17].

How about the the tachyon profile $T = \lambda \cosh(\mathbf{X}^0/\sqrt{2\alpha'})$ which was originally considered by Sen [17]? The computation of the partition function faces a problem that $T(\tau)T(0)$ diverges as $\tau \to 0$ because of contributions from cross terms, that is, $\langle : e^{X^0/\sqrt{2\alpha'}}(\tau) :: e^{-X^0/\sqrt{2\alpha'}}(0) : \rangle$. Due to this divergence, the integration over τ does not converge. However, now we have a BSFT action using the analytic continuation and it should be valid for this tachyon profile. In fact, if we notice that the contributions from the cross terms are proportional to the equation of motions and thus vanish, we can easily check that the pressure and energy computed from the action agree with those computed by Sen using the boundary state [17] to the order we considered.

B. Background constant field strength

Though the two-point functions obtained in this paper exhibit interesting higher derivative structures, at least around the on-shell momenta we can make a field redefinition to make them to be in a canonical form. In this sense any nontrivial consequence on effective action, except for the mass-shell conditions, may not come out from the two-point functions. However, if we include nontrivial backgrounds and compute two-point functions in those backgrounds, they contain information of higher point functions in a reduced manner. In this appendix, we follow this line and adopt a constant gauge field strength $\bar{F}_{\mu\nu}$ as a background. Many σ model calculations have been done so far on this background, which would be good for comparison with our method.

According to [6], the worldsheet boundary propagator is

$$\begin{aligned}
\langle X^\mu(\varphi_1) X^\nu(\varphi_2)\rangle &= 4\sum_{n=1}^{\infty}\frac{1}{n}e^{-\epsilon n}\left(G^{\mu\nu}\cos(n\varphi_{12}) - iH^{\mu\nu}\sin(n\varphi_{12})\right) \\
&= -2\left[(G^{\mu\nu} - H^{\mu\nu})\log(1 - e^{i\varphi_{12}-\epsilon}) + (G^{\mu\nu} + H^{\mu\nu})\log(1 - e^{-i\varphi_{12}-\epsilon})\right] , \\
\langle \psi^\mu(\varphi_1)\psi^\nu(\varphi_2)\rangle &= \frac{i}{2}\left[(H^{\mu\nu} - G^{\mu\nu})\frac{e^{(i\varphi_{12}-\epsilon)/2}}{1 - e^{i\varphi_{12}-\epsilon}} + (H^{\mu\nu} + G^{\mu\nu})\frac{e^{(-i\varphi_{12}-\epsilon)/2}}{1 - e^{-i\varphi_{12}-\epsilon}}\right]
\end{aligned} \tag{B.1}$$

where

$$G^{\mu\nu} = (1 - \bar{F}^2)^{-1} , \quad H^{\mu\nu} = \bar{F}(1 - \bar{F}^2)^{-1} . \tag{B.2}$$

Here $\bar{F}$ is a background constant field strength, and $G_{\mu\nu}$ is so-called open string metric. Let us first consider a self-contraction to give a renormalized boundary coupling. The boundary coupling is the same as before, (2.33). The redefinition of the gauge field to absorb the divergent factor coming from the normal ordering is

$$a_\mu(k) = a_{\mathrm{R}\mu(k)} \exp[2k_\mu G^{\mu\nu} k_\nu \log\epsilon] \ . \tag{B.3}$$

As in section 2.4, expansion of this and addition of a total derivative term gives

$$\begin{aligned} a_\rho(k) &= a_{\mathrm{R}\rho(k)} + 2k_\mu G^{\mu\nu} k_\nu (\log\epsilon) a_{\mathrm{R}\rho(k)} - 2k_\mu G^{\mu\nu} k_\rho (\log\epsilon) a_{\mathrm{R}\nu(k)} \\ &= a_{\mathrm{R}\rho(k)} - 2ik_\mu G^{\mu\nu} (\log\epsilon) f_{\mathrm{R}\nu\rho}(k) \ . \end{aligned} \tag{B.4}$$

This renormalization is the same as that of [6].

A crucial difference from the boundary coupling in section 2.4 appears in the self-contractions. The term (2.36), which vanished for the trivial background, is now giving a nonzero contribution:

$$\int d\tau \langle \dot{X}^\mu(\tau) X^\sigma(\tau)\rangle ik^\sigma : e^{ik\cdot\hat{X}} := \frac{1}{e^\epsilon - 1} 4H^{\mu\sigma} ik_\sigma \int d\tau : e^{ik\cdot\hat{X}} : \ . \tag{B.5}$$

In addition, the fermion self-contraction is also non-vanishing as

$$\langle \psi^\mu(\varphi)\psi^\nu(\varphi)\rangle = iH^{\mu\nu} \frac{e^{\epsilon/2}}{e^\epsilon - 1} \ . \tag{B.6}$$

Thus the diverging part of the self-contraction cancels with each other, while a finite term remains :

$$I_{\mathrm{B}} = -i\int d\tau \int dk \left(a_{\mathrm{R}\mu}(k) : \dot{X}^\mu e^{ik\cdot X}: -2 f_{\mathrm{R}\mu\nu}(k) : \psi^\mu \psi^\nu e^{ik\cdot X} : -iH^{\mu\nu} f_{\mathrm{R}\mu\nu}(k) : e^{ik\cdot X} : \right) \ .$$

The last term is the finite modification due to the renormalization in the presence of the background constant field strength.

Interestingly, this modification results in non-vanishing one-point function,

$$\langle I_{\mathrm{B}}\rangle = -H^{\mu\nu} F_{\mathrm{R}\mu\nu}(x) \ . \tag{B.7}$$

This is expected, as an expansion of the Maxwell Lagrangian around a constant field strength.[‡‡] So our renormalized boundary coupling is consistent with the usual target space picture.

Let us compute the two-point function with this boundary coupling. A straightforward calculation shows that

$$\langle I_{\mathrm{B}} I_{\mathrm{B}}\rangle = \int dk d\tilde{k}\ e^{i(k+\tilde{k})\cdot x} 8\pi^{3/2} 2^{4k_\mu G^{\mu\nu}\tilde{k}_\nu} \frac{\Gamma(2k_\mu G^{\mu\nu}\tilde{k}_\nu + 1/2)}{\Gamma(2k_\mu G^{\mu\nu}\tilde{k}_\nu + 1)} f_{\rho\sigma}(k) f_{\delta\gamma}(\tilde{k}) G^{\rho\delta} G^{\sigma\gamma} \ . \tag{B.8}$$

[‡‡] This existence of the one-point function does not cause any problem as opposed to the situation in section 2.5.2, because it is proportional to the field strength and thus it does not change the vacuum and the constant field strength is a solution of the equations of motion.

The higher-derivative part turns out eventually to be nothing different from the one obtained in the case of vanishing field strength background, except that the metric is now replaced by the open string metric $G_{\mu\nu}$. Our result (B.8) is consistent with the partition function results of [6]. However, our result is slightly different from the effective action derived from string scattering amplitudes provided in [6]. This discrepancy might be resolved by some field redefinitions or Jacobi-like identities among field strengths.

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